YANPEI LIU

INTRODUCTORY MAP THEORY

Kapa & Omega, Glendale, AZ

USA

2010

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Preface

Maps as a mathematical topic arose probably from the four color problem[Bir1, Ore1] and the more general map coloring problem[HiC1, Rin1, Liu11] in the mid of nineteenth century although maps as polyhedra which go back to the Platonic age. I could not list references in detail on them because it is well known for a large range of readers and beyond the scope of this book. Here, I only intend to present a comprehensive theory of maps as a rigorous mathematical concept which has been developed mostly in the last half a century.

However, as described in the book[Liu15] maps can be seen as graphs in development from partition to permutation and as a basis extended to Smarandache geometry shown in [Mao3–4]. This is why maps are much concerned with abstraction in the present stage.

In the beginning, maps as polyhedra were as a topological, or geometric object even with geographical consideration[Kem1]. The first formal definition of a map was done by Heffter from [Hef1] in the 19th century. However, it was not paid an attention by mathematicians until 1960 when Edmonds published a note in the AMS Notices with the dual form of Heffter's in [Edm1,Liu3].

Although this concept was widely used in literature as [Liu1–2, Liu4–6, Rin1–3, Sta1–2, *et al*], its disadvantage for the nonorientable case involved does not bring with some convenience for clarifying some related mathematical thinking.

Since Tutte described the nonorientability in a new way [Tut1–3], a number of authors begin to develop it in combinatorization of continuous objects as in [Lit1, Liu7–10, Vin1–2, *et al*].

The above representations are all with complication in constructing an embedding, or all distinct embeddings of a graph on a surface.

However, the joint tree model of an embedding completed in recent years and initiated from the early articles at the end of seventies in the last century by the present author as shown in [Liu1–2] enables us to make the complication much simpler.

Because of the generality that an asymmetric object can always be seen with some local symmetry in certain extent, the concepts of graphs and maps are just put in such a rule. In fact, the former is corresponding to that a group of two elements sticks on an edge and the later is that a group of four elements sticks on an edge such that a graph without symmetry at all is in company with local symmetry. This treatment will bring more advantages for observing the structure of a graph. Of course, the later is with restriction of the former because of the later as a permutation and the former as a partition.

The joint tree representation of an embedding of a graph on two dimensional manifolds, particularly surfaces(compact 2-manifolds without boundary in our case), is described in Chapter I for simplifying a number of results old and new.

This book contains the following chapters in company with related subjects.

In Chapter I, the embedding of a graph on surfaces are much concerned because they are motivated to building up the theory of abstract maps related with Smarandache geometry.

The second chapter is for the formal definition of abstract maps. One can see that this matter is a natural generalization of graph embedding on surfaces.

The third chapter is on the duality not only for maps themselves but also for operations on maps from one surface to another. One can see how the duality is naturally deduced from the abstract maps described in the second chapter.

The fourth chapter is on the orientability. One can see how the orientability is formally designed as a combinatorial invariant. The fifth chapter concentrates on the classification of orientable maps. The sixth chapter is for the classification of nonorientable maps.

From the two chapters: Chapter V and Chapter VI, one can see

how the procedure is simplified for these classifications.

The seventh chapter is on the isomorphisms of maps and provides an efficient algorithm for the justification and recognition of the isomorphism of two maps, which has been shown to be useful for determining the automorphism group of a map in the eighth chapter. Moreover, it enables us to access an automorphism of a graph.

The ninth and the tenth chapters observe the number of distinct asymmetric maps with the size as a parameter. In the former, only one vertex maps are counted by favorite formulas and in the latter, general maps are counted from differential equations. More progresses about this kind of counting are referred to read the recent book[Liu7] and many further articles[Bax1, BeG1, CaL1–2, ReL1–3, *etc*].

The next chapter, Chapter XI, only presents some ideas for accessing the symmetric census of maps and further, of graphs. This topic is being developed in some other directions[KwL1–2] and left as a subject written in the near future.

From Chapter XII through Chapter XV, extensions from basic theory are much concerned with further applications.

Chapter XII discusses in brief on genus polynomial of a graph and all its super maps rooted and unrooted on the basis of the joint tree model. Recent progresses on this aspect are referred to read the articles [Liu13–15, LiP1, WaL1–2, ZhL1–2, ZuL1, *etc*].

Chapter XIII is on the census of maps with vertex or face partitions. Although such census involves with much complication and difficulty, because of the recent progress on a basic topic about trees via an elementary method firstly used by the author himself we are able to do a number of types of such census in very simple way. This chapter reflects on such aspects around.

Chapter XIV is on graphs that their super maps are particularly considered for asymmetrical and symmetrical census via their semi-automorphism and automorphism groups or via embeddings of graphs given [Liu19, MaL1, MaT1, MaW1, *etc*].

Chapter XV, is on functional equations discovered in the census of a variety of maps on sphere and general surfaces. Although their

well definedness has been done, almost all of them have not yet been solved up to now.

Three appendices are compliment to the context. One provides the clarification of the concepts of polyhedra, surfaces, embeddings, and maps and their relationship. The other two are for exhaustively calculating numerical results and listing all rooted and unrooted maps for small graphs with more calculating results compared with those appearing in [Liu14], [Liu17] and [Liu19].

From a large amount of materials, more than hundred observations for beginners probably senior undergraduates, more than hundred exercises for mainly graduates of master degree and more than hundred research problems for mainly graduates of doctoral degree are carefully designed at the end of each chapter in adapting the needs of such a wide range of readers for mastering, extending and investigating a number of ways to get further development on the basic theory of abstract maps.

Although I have been trying to design this book self contained as much as possible, some books such as [DiM1], [Mss1] and [GaJ1] might be helpful to those not familiar with basic knowledge of permutation groups, topology and computing complexity as background.

Since early nineties of the last century, a number of my former and present graduates were or are engaged with topics related to this book. Among them, I have to mention Dr. Ying Liu[LpL1], Dr. Yuanqiu Huang[HuL1], Dr. Junliang Cai[CaL1–2], Dr. Deming Li[LiL1], Dr. Han Ren[ReL1–3], Dr. Rongxia Hao[HaC1, HaL1], Dr. Zhaoxiang Li[LiQ1–2], Dr. Linfan Mao[MaL1, MaT1, MaW1], Dr. Erling Wei[WiL1–2], Dr. Weili He[HeL1], Dr. Liangxia Wan[WaL1–2], Dr. Yichao Chen[CnL1, CnR1], Dr. Yan Xu[XuL1–2], Dr. Wenzhong Liu[LwL1–2], Dr. Zeling Shao[ShL1], Dr. Yan Yang[YaL1–2], Dr. Guanghua Dong[DoL1], Ms. Ximei Zhao[ZhL1–2], Mr. Lifeng Li[LiP1], Ms. Huiyan Wang[WgL1], Ms. Zhao Chai[CiL1], Mr. Zilong Zhu[ZuL1], *et al* for their successful work related to this book.

On this occasion, I should express my heartiest appreciation of the financial support by KOSEF of Korea from the Com^{2}MaC (Com-

binatorial and Computational Mathematics Research Center) of the Pohang University of Science and Technology in the summer of 2001. In that period, the intention of this book was established. Moreover, I should be also appreciated to the Natural Science Foundation of China for the research development reflected in this book under its Grants(60373030, 10571013, 10871021).

Y.P. Liu
Beijing, China
Jan., 2010

Contents

Chapter I

Abstract Embeddings

- A graph is considered as a partition on the union of sets obtained from each element of a given set the binary group $B = \{0, 1\}$ sticks on.

- A surface, *i.e.*, a compact 2-manifold without boundary in topology, is seen as a polygon of even edges pairwise identified.

- An embedding of a graph on a surface is represented by a joint tree of the graph. A joint of a graph consists of a plane extended tree with labelled cotree semi-edges. Two semi-edges of a cotree edge has the same label as the cotree edge with a binary index. An extended tree is compounded of a spanning tree with cotree semi-edges.

- Combinatorial properties of an embedding in abstraction are particularly discussed for the formal definition of a map.

I.1 Graphs and networks

Let X be a finite set. For any $x \in X$, the binary group $B = \{0, 1\}$ sticks on x to obtain $Bx = \{x(0), x(1)\}$. $x(0)$ and $x(1)$ are called the *ends* of x, or Bx. If Bx is seen as an ordered set $\langle x(0), x(1)\rangle$, then

$x(0)$ and $x(1)$ are, respectively,*initial* and *terminal* ends of x. Let

$$\mathcal{X} = \sum_{x \in X} Bx, \tag{1.1}$$

i.e., the disjoint union of all Bx, $x \in X$. $\mathcal{X}$ is called the *ground set*.

A (*directed*) *pregraph* is a *partition* Par= $\{P_1, P_2, \cdots\}$ of the ground set $\mathcal{X}$,*i.e.*,

$$\mathcal{X} = \sum_{i \geq 1} P_i. \tag{1.2}$$

Bx (or $\langle x(0), x(1)\rangle$), or simply denoted by x itself, $x \in X$, is called an (*arc*) *edge* and P_i, $i \geq 1$, a *node* or *vertex*.

A (directed) pregraph is written as $G = (V, E)$ where V =Par and

$$E = B(X) = \{Bx | x \in X\}$$

$$(= \{\langle x(0), x(1)\rangle | x \in X\}).$$

If X is a finite set, the (directed) pregraph is called *finite*; otherwise, *infinite*. In this book, (directed) pregraphs are all finite.

If $X = \emptyset$, then the (directed) pregraph is said to be *empty* as well.

An edge (arc) is considered to have two semiedges each of them is incident with only one end (semiarcs with directions of one from the end and the other to the end). An edge (arc) is with two ends identified is called a selfloop (di-selfloop); otherwise, a link (di-link). If t edges (arcs) have same ends (same direction) are called a multiedge (multiarc), or t-edge (t-arc).

Example 1.1 There are two directed pregraphs on $X = \{x\}$, *i.e.*,

$$\text{Par}_1 = \{\{x(0)\}, \{x(1)\}\};$$

$$\text{Par}_2 = \{\{x(0), x(1)\}\}.$$

They are all distinct pregraphs as well as shown in Fig.1.1.

Par_1 Par_2

Fig.1.1 Directed pregraphs of 1 edge

Further, pregraphs of size 2 are observed.

Example 1.2 On $X = \{x_1, x_2\}$, the 15 directed pregraphs are as follows:

$$
\begin{aligned}
&\mathrm{Par}_1 = \{\{x_1(0)\}, \{x_1(1)\}, \{x_2(0)\}, \{x_2(1)\}\};\\
&\mathrm{Par}_2 = \{\{x_1(0), x_1(1)\}, \{x_2(0)\}, \{x_2(1)\}\};\\
&\mathrm{Par}_3 = \{\{x_1(0), x_2(0)\}, \{x_1(1)\}, \{x_2(1)\}\};\\
&\mathrm{Par}_4 = \{\{x_1(0), x_2(1)\}, \{x_1(1)\}, \{x_2(0)\}\};\\
&\mathrm{Par}_5 = \{\{x_1(0)\}, \{x_1(1), x_2(0)\}, \{x_2(1)\}\};\\
&\mathrm{Par}_6 = \{\{x_1(0)\}, \{x_1(1), x_2(1)\}, \{x_2(0)\}\};\\
&\mathrm{Par}_7 = \{\{x_1(0)\}, \{x_1(1)\}, \{x_2(1), x_2(0)\}\};\\
&\mathrm{Par}_8 = \{\{x_1(0), x_1(1), x_2(0)\}, \{x_2(1)\}\};\\
&\mathrm{Par}_9 = \{\{x_1(0), x_1(1), x_2(1)\}, \{x_2(0)\}\};\\
&\mathrm{Par}_{10} = \{\{x_1(0), x_2(0), x_2(1)\}, \{x_1(1)\}\};\\
&\mathrm{Par}_{11} = \{\{x_1(0)\}, \{x_1(1), x_2(0), x_2(1)\}\};\\
&\mathrm{Par}_{12} = \{\{x_1(0), x_1(1), x_2(0), x_2(1)\}\};\\
&\mathrm{Par}_{13} = \{\{x_1(0), x_1(1)\}, \{x_2(0), x_2(1)\}\};\\
&\mathrm{Par}_{14} = \{\{x_1(0), x_2(0)\}, \{x_1(1), x_2(1)\}\};\\
&\mathrm{Par}_{15} = \{\{x_1(0), x_2(1)\}, \{x_1(1), x_2(0)\}\}.
\end{aligned}
$$

Among the 15 directed pregraphs, Par_3, Par_4, Par_5 and Par_6 are 1 pregraph; Par_8 and Par_9 are 1 pregraph; Par_{10} and Par_{11} are 1 pregraph; Par_{14} and Par_{15} are 1 pregraph; and others are 1 pregraph each. Thus, there are 9 pregraphs in all(as shown in Fig.1.2).

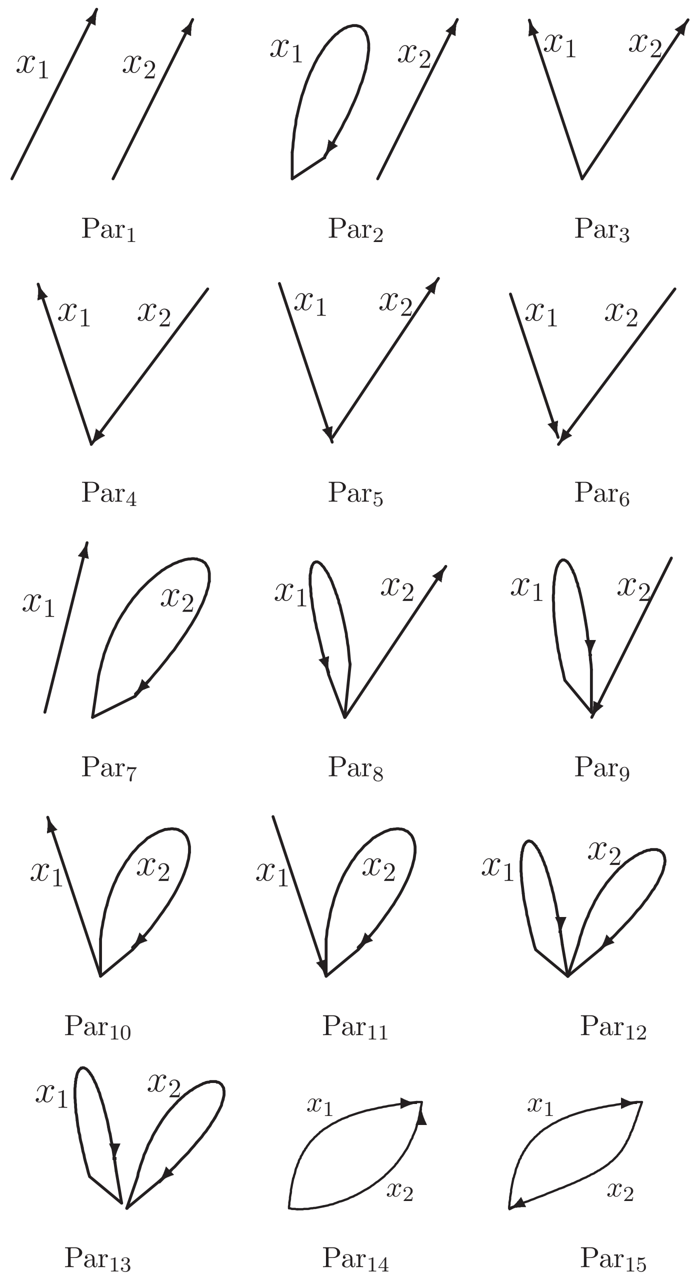


Fig.1.2 Directed pregraphs of 2 edges

Now, Par$= \{P_1, P_2, \cdots\}$ and $\mathcal{B}$ are, respectively, seen as a mapping $z \mapsto P_i$, $z \in P_i$, $i \geq 1$ and a mapping $z \mapsto \bar{z}$, $\bar{z} \neq z$, $\{z, \bar{z}\} \in B(X)$. The *composition* of two mappings α and β on a set $\mathcal{Z}$ is defined

to be the mapping

$$(\alpha\beta)z = \bigcup_{y\in\beta z} \alpha y,\ \ z \in \mathcal{Z}. \tag{1.3}$$

Let $\Psi_{\{\mathrm{Par},\mathcal{B}\}}$ be the semigroup generated by Par=Par(X) and $\mathcal{B} = B(X)$. Since the mappings α =Par and $\mathcal{B}$ have the property that $y \in \alpha z \Leftrightarrow z \in \alpha y$, it can be checked that for any $z, y \in B(X)$, what is determined by

$$\exists\gamma \in \Psi_{\{\mathrm{Par},\mathcal{B}\}},\ \ z \in \gamma y$$

is an equivalence. If $B(X)$ itself is a equivalent class, then the semigroup $\Psi_{\{\mathrm{Par},\mathcal{B}\}}$ is called *transitive* on $\mathcal{X} = B(X)$. A (directed)pregraph with $\Psi_{\{\mathrm{Par},\mathcal{B}\}}$ transitive on $\mathcal{X}$ is called a (*directed*)*graph* .

A (directed)pregraph $G = (V, E)$ that for any two vertices $u, v \in V$, there exists a sequence of edges $e_1, e_2, \cdots, e_s$ for the two ends of e_i, $i = 2, 3, \cdots, s-1$, are in common with those of respective e_{i-1} and e_{i+1} where u and v are, respectively, the other ends of e_1 and e_s, is called *connected* . Such a sequence of edges is called a *trail* between u and v. A trail without edge repetition is a *walk*. A walk without vertex repetition is a *path*. A trail, walk, or path with $u = v$ is, respectively, a *travel*, *tour*, or *circuit*.

Theorem 1.1 A (directed)pregraph is a (directed)graph if, and only if, it is connected.

Proof Necessity. Since $\mathrm{Par}^k = \mathrm{Par}$, $k \geq 1$, and $\mathcal{B}^k = \mathcal{B}$, $k \geq 1$, by the transitivity, for any two elements $y, z \in \mathcal{X}$, there exists γ such that $z \in \gamma y$ and there exists an integer $n \geq 0$ such that

$$\gamma = (\mathcal{B}\mathrm{Par})^n\mathcal{B} = \underbrace{(\mathcal{B}\mathrm{Par})\cdots(\mathcal{B}\mathrm{Par})}_{n}\mathcal{B}, \tag{1.4}$$

where $\mathcal{B}$Par appears for n times. Therefore, the (directed)pregraph is connected.

Sufficiency. If a (directed)pregraph is connected, *i.e.*, for any two elements $x, y \in \mathcal{X}$, their incident vertices $u, v \in V$, have edges $e_1, e_2, \cdots, e_s$, such that e_i, $i = 2, 3, \cdots, s-1$, is in common with e_{i-1}

and e_{i+1}. Of course, u and v are, respectively, the ends of e_1 and e_s. Thus, $y \in \gamma z$ where$\gamma = (\mathrm{Par}\mathcal{B})^s\mathcal{B}$. This implies that the semigroup $\Psi_{\{\mathrm{Par},\mathcal{B}\}}$ is transitive on $\mathcal{X}$. Therefore, the (directed)pregraph is a (directed)graph. □

It is seen from the theorem that (directed) graphs here are, in fact, connected (directed) graphs in most textbooks. Because disconnectedness is rarely necessary to consider, for convenience all graphs, embeddings and then maps in what follows are defined within connectedness in this book.

A *network* N is such a graph $G = (V, E)$ with a real function $w(e) \in \mathbf{R}, e \in E$ on E, and hence write $N = (G; w)$. Usually, a network N is denoted by the graph G itself if no confusion occurs.

Finite recursion principle On a finite set A, choose $a_0 \in A$ as the initial element at the 0th step. Assume a_i is chosen at the ith, $i \geq 0$, step with a given rule. If not all elements available from a_i are not yet chosen, choose one of them as a_{i+1} at the $i+1$st step by the rule, then a chosen element will be encountered in finite steps unless all elements of A are chosen.

Finite restrict recursion principle On a finite set A, choose $a_0 \in A$ as the initial element at the 0th step. Assume a_i is chosen at the ith, $i \geq 0$, step with a given rule. If a_0 is not available from a_i, choose one of elements available from a_i as a_{i+1} at the $i+1$st step by the rule, then a_0 will be encountered in finite steps unless all elements of A are chosen.

The two principles above are very useful in finite sets, graphs and networks, even in a wide range of combinatorial optimizations.

A $G = (V, E)$ with $V = V_1 + V_2$ of both V_1 and V_2 independent, *i.e.*, its vertex set is partitioned into two parts with each part having no pair of vertices adjacent, is called *bipartite*.

Theorem 1.2 A graph $G = (V, E)$ is bipartite if, and only if, G has no circuit with odd number of edges.

Proof Necessity. Since G is bipartite, start from $v_0 \in V$ ini-

tially chosen and then by the rule from the vertex just chosen to one of its adjacent vertices via an edge unused and then marked by used, according to the finite recursion principle, an even circuit (from bipartite), or no circuit at all, can be found. From the arbitrariness of v_0 and the way going on, no circuit of G is with odd number of edges.

Sufficiency. Since all circuits are even, start from marking an arbitrary vertex by 0 and then by the rule from a vertex marked by $b \in B = \{0, 1\}$ to mark all its adjacent vertices by $\overline{b} = 1 - b$, according to the finite recursion principle the vertex set is partitioned into $V_0 = \{v \in V |$ marked by $0\}$ and $V_1 = \{v \in V |$ marked by $1\}$. By the rule, V_0 and V_1 are both independent and hence G is bipartite. □

From this theorem, a graph without circuit is bipartite. In fact, from the transitivity, any graph without circuit is a tree.

On a pregraph, the number of elements incident to a vertex is called the *degree* of the vertex. A pregraph of all vertices with even degree is said to be *even* . If an even pregraph is a graph, then it is called a *Euler graph*.

Theorem 1.3 A pregraph $G = (V, E)$ is even if, and only if, there exist circuits $C_1, C_2, \cdots, C_n$, on G such that

$$E = C_1 + C_2 + \cdots + C_n, \tag{1.5}$$

where n is a nonnegative integer.

Proof Necessity. Since all the degrees of vertices on G are even, any pregraph obtained by deleting the edges of a circuit from G is still even. From the finite recursion principle, there exist a nonnegative integer n and circuits $C_1, C_2, \cdots, C_n$, on G such that (1.5) is satisfied.

Sufficiency. Because a circuit contributes 2 to the degree of each of its incident vertices, (1.5) guarantees each of vertices on G has even degree. Hence, G is even. □

The set of circuits $\{C_i | 1 \leq i \leq n\}$ of G in (1.5) is called a *circuit partition*, or written as Cir=Cir(G). Two direct conclusions of Theorem 1.3 are very useful . One is the case that G is a graph. The

other is for G is a directed pregraph. Their forms and proofs are left for the reader.

Let $N = (G; w)$ be a network where $G = (V, E)$ and $w(e) = -w(e) \in Z_n = \{0, 1, \cdots, n-1\}$, *i.e.*, mod n, $n \geq 1$, integer group. For examples, $Z_1 = \{0\}$, $Z_2 = B = \{0, 1\}$ *etc.* Suppose $x_v = -x_v \in Z_n$, $v \in V$, are variables. Let us discuss the system of equations

$$x_u + x_v = w(e) \pmod n, \ e = (u, v) \in E \tag{1.6}$$

on Z_n.

Theorem 1.4 System of equations(1.6) has a solution on Z_n if, and only if, there is no circuit C such that

$$\sum_{e \in C} w(e) \neq 0 \pmod n \tag{1.7}$$

on N.

Proof Necessity. Assume C is a circuit satisfying (1.7) on N. Because the restricted part of (1.6) on C has no solution, the whole system of equations (1.6) has to be no solution either. Therefore, N has no such circuit. This is a contradiction to the assumption

Sufficiency. Let $x_0 = a \in Z_n$, start from $v_0 \in V$ reached. Assume $v_i \in V$ reached and $x_i = a_i$ at step i. Choose one of $e_i = (v_i, v_{i+1}) \in E$ without used(otherwise, backward 1 step as the step i). Choose v_{i+1} reached and e_i used with $a_{i+1} = a_i + w(e_i)$ at step $i+1$. If a circuit as $\{e_0, e_1, \cdots, e_l\}$, $e_j = (v_j, v_{j+1}), 0 \leq j \leq l, v_{l+1} = v_0$, occurs within a permutation of indices, then from (1.7)

$$\begin{aligned}
a_{l+1} &= a_l + w(e_l) \\
&= a_{l-1} + w(e_{l-1}) + w(e_l) \\
&\cdots\cdots \\
&= a_0 + \sum_{j=0}^{l} w(e_j) \\
&= a_0.
\end{aligned}$$

Because the system of equations obtained by deleting all the equations for all the edges on the circuit from (1.6) is equivalent to the original system of equations (1.6), in virtue of the finite recursion principle a solution of (1.6) can always be extracted . □

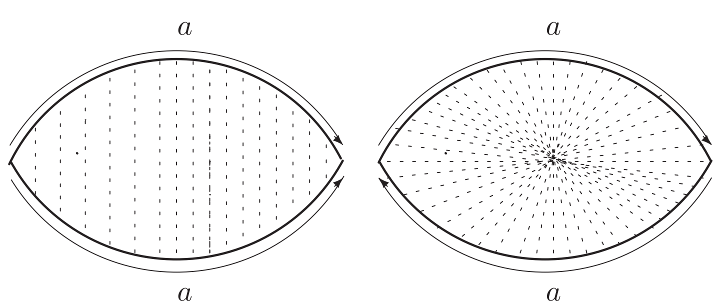
When $n = 2$, this theorem has a variety of applications. In [Liu5], some applications can be seen. Further, its extension on a nonAbelian group can also be done while the system of equations are not yet linear but quadratic.

I.2 Surfaces

In topology, a surface is a compact 2-dimensional manifold without boundary. In fact, it can be seen as what is obtained by identifying each pair of edges on a polygon of even edges pairwise.

For example, in Fig.1.3, two ends of each dot line are the same point. The left is a sphere, or the plane when the infinity is seen as a point. The right is the projective plane. From the symmetry of the sphere, a surface can also seen as a sphere cutting off a polygon with pairwise edges identified.

The two surfaces in Fig.1.3 are formed by a polygon of two edges pairwise as a.



Sphere(Plane) Projective plane

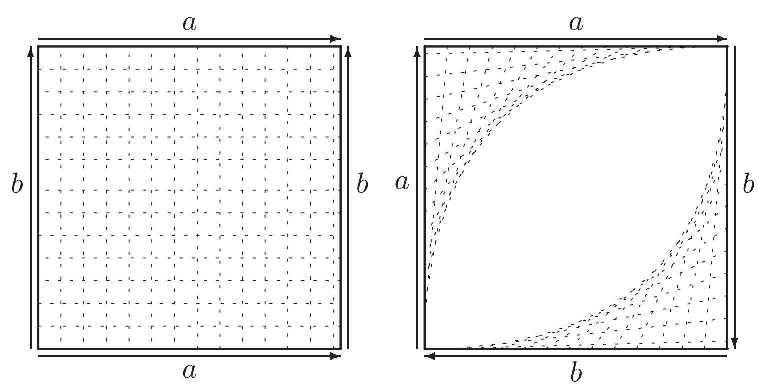
Fig.1.3 Sphere and projective plane

Surface closed curve axiom A closed curve on a surface has one of the two possibilities: one side and two sides.

A curve with two sides is called a *double side curve* ; otherwise, a *single side curve* . As shown in Fig.1.3, any closed curve on a sphere is a double side curve(In fact, this is the Jordan curve axiom). However, it is different from the sphere for the projective plane. there are both a single(shown by a dot line) and a double side curve.

How do we justify whether a closed curve on a surface is of single side, or not?

In order to answer this question, the concept of contractibility of a curve has to be clarified. If a closed curve on a surface can be continuously contracted along one side into a point, then it is said to be *contractible*, or *homotopic* to 0.



Torus Klein bottle

Fig.1.4 Torus and Klein bottle

It is seen that a single side curve is never homotopic to 0 and a double side curve is not always homotopic to 0. For example, in Fig.1.4, the left, *i.e.*, the torus, each of the dot lines is of double side but not contractible. The right, *i.e.*, the Klein bottle, all the dot lines are of single side , and hence, none of them is contractible.

A surface with all closed curves of double side is called *orientable*; otherwise, *nonorientable* .

For example, in Fig.1.3, the sphere is orientable and the projec-

tive plane is nonorientable. In Fig.1.4, the torus is orientable and the Klein bottle is nonorientable.

The maximum number of closed curves cutting along without destroying the continuity on a surface is called the *pregenus* of the surface.

In view of Jordan curve axiom, there is no such closed curve on the sphere. Thus, the pregenus of sphere is 0. On the projective plane, only one such curve is available (each of dot lines is such a closed curve in Fig.1.3) and hence the pregenus of projective plane is 1.

Similarly, the pregenera of torus and Klein bottle are both 2 as shown in Fig.1.4.

Theorem 1.5 The pregenus of an orientable surface is a non-negative even number.

A formal proof can not be done until Chapter 5. Based on this theorem, the genus of an orientable surface can be defined to be half its pregenus, called the *orientable genus*. The genus of a nonorientable surface, called *nonorientable genus*, is its pregenus itself.

The sphere is written as aa^{-1} where a^{-1} is with the opposite direction of a on the boundary of the polygon. Thus, the projective plane, torus and Klein bottle are, respectively, aa, $aba^{-1}b^{-1}$ and $aabb$. In general,

$$\begin{aligned} O_p &= \prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1} \\ &= a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_p b_p a_p^{-1} b_p^{-1} \end{aligned} \tag{1.8}$$

and

$$Q_q = \prod_{i=1}^{q} a_i a_i = a_1 a_1 a_2 a_2 \cdots a_q a_q \tag{1.9}$$

denote, respectively, a surface of orientable genus p and a surface of nonorientable genus q. Of course, O_0, Q_1, O_1 and Q_2 are, respectively, the sphere, projective plane, torus and Klein bottle.

It is easily checked that whenever an even polygon is with a pair of its edges in the same direction, the polygon represents a nonori-

entable surface. Thus, all O_p, $p \geq 0$, orientable and all Q_q, $q \geq 1$, are nonorientable.

Forms (1.8) and (1.9) are said to be *standard*.

If the *form* of a surface is defined by its orientability and its genus, then the operations 1–3 and their inverses shown as in Fig.1.5–7, do not change the surface form.

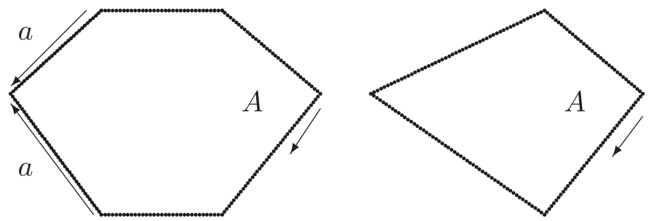


Fig.1.5 Operation 1: $Aaa^{-1} \Longleftrightarrow A$

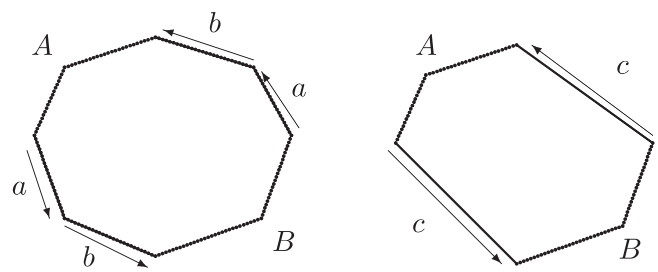


Fig.1.6 Operation 2: $AabBab \Longleftrightarrow AcBc$

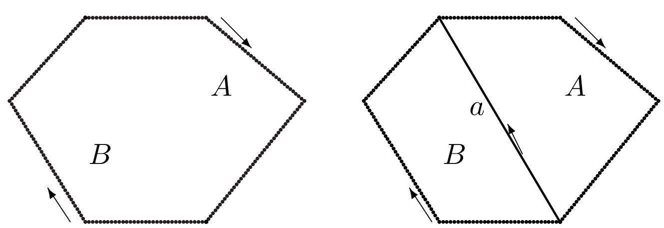


Fig.1.7 Operation 3: $AB \Longleftrightarrow (Aa)(a^{-1}B)$

In fact, what is determined under these operations is just a topological equivalence, denoted by $\sim_{\rm top}$.

Notice that A and B are all linear order of letters and permitted

to be empty as degenerate case in these operations.

The parentheses stand for cyclic order when more than one cyclic orders occur for distinguishing from one to another.

Relation 0 On a surface (A, B), if A is a surface itself then $(A, B) = ((A)x(B)x^{-1}) = ((A)(B))$.

Relation 1 $(AxByCx^{-1}Dy^{-1}) \sim_{\text{top}} ((ADCB)(xyx^{-1}y^{-1}))$.

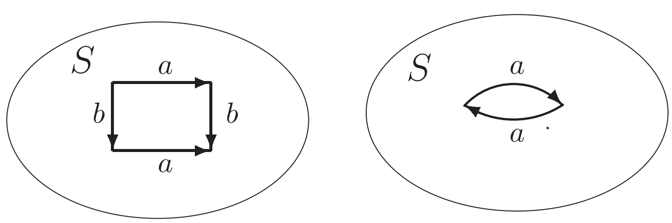
Relation 2 $(AxBx) \sim_{\text{top}} ((AB^{-1})(xx))$.

Relation 3 $(Axxyzy^{-1}z^{-1}) \sim_{\text{top}} ((A)(xx)(yy)(zz))$.

In the four relations, A, B, C, and D are permitted to be empty. $B^{-1} = b_s^{-1} \cdots b_3^{-1} b_2^{-1} b_1^{-1}$ is also called the inverse of $B = b_1 b_2 b_3 \cdots b_s$, $s \geq 1$. Parentheses are always omitted when unnecessary to distinguish cyclic or linear order.

On a surface S, the operation of cutting off a quadrangle $aba^{-1}b^{-1}$ and then identifying each pair of edges with the same letter is called a *handle* as shown in the left of Fig.1.8.

If the quadrangle $aba^{-1}b^{-1}$ is replaced by aa, then such an operation is called a *crosscap* as shown in the right of Fig.1.8.



Handle Crosscap

Fig.1.8 Handle and crosscap

The following theorem shows the result of doing a handle on an orientable surface.

Theorem 1.6 What is obtained by doing a handle on an orientable surface is still orientable with its genus 1 added.

Proof Suppose S is the surface obtained, then

$$\begin{aligned} S &\sim_{\text{top}} \prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1} x a_{p+1} b_{p+1} a_{p+1}^{-1} b_{p+1}^{-1} x^{-1} \text{ (Relation 0)} \\ &\sim_{\text{top}} \prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1} x x^{-1} a_{p+1} b_{p+1} a_{p+1}^{-1} b_{p+1}^{-1} \text{ (Relation 1)} \\ &\sim_{\text{top}} \prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1} a_{p+1} b_{p+1} a_{p+1}^{-1} b_{p+1}^{-1} \text{ (Operation 1)} \\ &= \prod_{i=1}^{p+1} a_i b_i a_i^{-1} b_i^{-1}. \end{aligned}$$

This is the theorem. □

In the above proof, x and x^{-1} are a line connecting the two boundaries to represent the surface as a polygon shown in Fig.1.8. This procedure can be seen as the degenerate case of operation 3.

In what follows, observe the result by doing a crosscap on an orientable surface.

Theorem 1.7 On an orientable surface of genus p, $p \geq 0$, what is obtained by doing a crosscap is nonorientable with its genus $2p+1$.

Proof Suppose N is the surface obtained, then

$$\begin{aligned} N &\sim_{\text{top}} \prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1} x a a x^{-1} \text{ (Relation 0)} \\ &\sim_{\text{top}} \prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1} x x^{-1} c_1 c_1 \text{ (Relation 2)} \\ &\sim_{\text{top}} \prod_{i=2}^{p} a_i b_i a_i^{-1} b_i^{-1} x x^{-1} c_1 c_1 c_2 c_2 c_3 c_3 \text{ (Relation 3)} \\ &\sim_{\text{top}} \prod_{i=1}^{2p+1} c_i c_i. \text{ (Relation 3 by } p-1 \text{ times).} \end{aligned}$$

This is the theorem. □

By doing a handle on a nonorientable surface, 2 more genus should be added with the same nonorientability.

Theorem 1.8 On a nonorientable surface, what is obtained by doing a handle is nonorientable with its genus 2 added.

Proof Suppose N is the obtained surface, then

$$\begin{aligned} N &\sim_{\text{top}} \prod_{i=1}^{q} a_i a_i x a b a^{-1} b^{-1} x^{-1} \text{ (Relation 0)} \\ &\sim_{\text{top}} \prod_{i=1}^{q} a_i a_i x x^{-1} a b a^{-1} b^{-1} \text{ (Relation 1)} \\ &\sim_{\text{top}} \prod_{i=1}^{q} a_i a_i a b a^{-1} b^{-1} \text{ (Operation 1)} \\ &\sim_{\text{top}} \prod_{i=1}^{q+2} c_i c_i. \text{ (Relation 3).} \end{aligned}$$

This is the theorem. □

By doing a crosscap on a nonorientable surface, 1 more genus produced with the same nonorientability

Theorem 1.9 On a nonorientable surface, what is obtained by doing a crosscap is nonorientable with its genus 1 added.

Proof Suppose N is the obtained surface, then

$$\begin{aligned} N &\sim_{\text{top}} \prod_{i=1}^{q} a_i a_i x a a x^{-1} \text{ (Relation 0)} \\ &\sim_{\text{top}} \prod_{i=1}^{q} a_i a_i x x^{-1} a a \text{ (Relation 2)} \\ &\sim_{\text{top}} \prod_{i=1}^{q} a_i a_i a a \text{ (Operation 1)} \\ &\sim_{\text{top}} \prod_{i=1}^{q+1} a_i a_i. \text{ (Relation 3).} \end{aligned}$$

This is the theorem. □

I.3 Embedding

Let $G = (V, E)$, $V = \{v_1, v_2, \cdots, v_n\}$, be a graph. A point in the 3-dimensional space is represented by a real number t as the parameter, *e.g.*, $(x, y, z) = (t, t^2, t^3)$. Write the vertices as

$$v_i = (x_i, y_i, z_i) = (t_i, t_i^2, t_i^3)$$

such that $t_i \neq t_j$, $i \neq j$, $1 \leq i, j \leq n$, and an edge as

$$(u, v) = u + \lambda v,\ 1 \leq \lambda \leq 1,$$

i.e., the straight line segment between u and v. Because for any four vertices v_i, v_j, v_l and v_k,

$$\det \begin{pmatrix} x_i - x_j & x_i - x_l & x_i - x_k \\ y_i - y_j & y_i - y_l & y_i - y_k \\ z_i - z_j & z_i - z_l & z_i - z_k \end{pmatrix}$$

$$= \det \begin{pmatrix} t_i - t_j & t_i - t_l & t_i - t_k \\ t_i^2 - t_j^2 & t_i^2 - t_l^2 & t_i^2 - t_k^2 \\ t_i^3 - t_j^3 & t_i^3 - t_l^3 & t_i^3 - t_k^3 \end{pmatrix}$$

$= (t_i - t_j)(t_i - t_l)(t_i - t_k)(t_k - t_l)(t_k - t_j)(t_j - t_l) \neq 0$, *i.e.*, the four points are not coplanar, any two edges in G has no intersection inner point.

A representation of a graph on a space with vertices as points and edges as curves pairwise no intersection inner point is called an *embedding* of the graph in the space. If all edges are straight line segments in an embedding, then it is called a *straight line embedding*. Thus, any graph has a straight line embedding in the 3-dimensional space. Similarly, A *surface embedding* of graph G is a continuous injection μG of an embedding of G on the 3-dimensional space to a surface S such that each connected component of $S - \mu G$ is homotopic to 0. The connected component is called a *face* of the embedding. In early books, a surface embedding is also called a *cellular embedding*. Because only a surface embedding is concerned with in what follows, an embedding is always meant a surface embedding if not necessary to specify.

A graph without circuit is called a *tree*. A *spanning tree* of a graph is such a subgraph that is a tree with the same order as the graph. Usually, a spanning tree of a graph is in short called a tree on the graph. For a tree on a graph, the numbers of edges on the tree and not on the tree are only dependent on the order of the graph. They are, respectively, called the *rank* and the *corank* of the graph. The corank is also called the *Betti number*, or *cyclic number* by some authors.

The following procedure can be used for finding an embedding on a surface.

First, given a cyclic order of all semiedges at each vertex of G, called a *rotation*. Find a tree(spanning, of course) T on G and distinguish all the edges not on T by letters. Then, replace each edge not of T by two articulate edges with the same letter.

From this procedure, G is transformed into $\tilde{G}$ without changing the rotation at each vertex except for new vertices that are all articulate. Because $\tilde{G}$ is a tree, according to the rotation, all lettered articulate edges of $\tilde{G}$ form a polygon with β pairs of edges, and hence

a surface in correspondence with a choice of indices on each pair of the same letter. For convenience, $\tilde{G}$ with a choice of indices of pair in the same letter is called a *joint tree* of G.

Theorem 1.10 A graph $G = (V, E)$ can always embedded into a surface of orientable genus at most $\lfloor \beta/2 \rfloor$, or of nonorientable genus at most β, where β is the Betti number of G.

Proof It is seen that any joint tree of G is an embedding of G on the surface determined by its associate polygon. From (1.8) for the orientable case, the surface has its genus at most $\lfloor 2\beta/4 \rfloor = \lfloor \beta/2 \rfloor$. From (1.9) for the nonorientable case, the surface has its genus at most $2\beta/2 = \beta$. □

In Fig.1.9, graph G and one of its joint tree are shown. Here, the spanning tree T is represented by edges without letter. a, b and c are edges not on T. Because the polygon is

$$\begin{aligned} abcacb &\sim_{\text{top}} c^{-1}b^{-1}cbaa \text{ (Relation 2)} \\ &\sim_{\text{top}} aabbcc \text{ (Theorem 1.7)}, \end{aligned}$$

the joint tree is, in fact, an embedding of G on a nonorientable surface of genus 3.

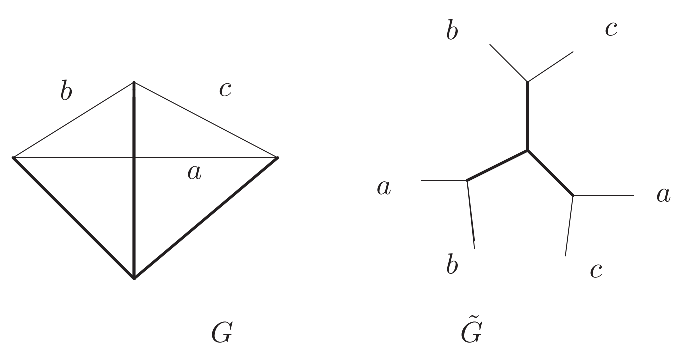


Fig.1.9 Graph and its joint tree

Because any graph with given rotation can always immersed in the plane in agreement with the rotation, each edge has two sides. As known, embeddings of a graph on surfaces are distinguished by the rotation of semiedges at each vertex and the choice of indices of the two semiedges on each edge of the graph whenever edges are labelled

by letters. *Different indices* of the two semiedges of an edge stand for from one side of the edge to the other on a face boundary in an embedding.

Theorem 1.11 A tree can only be embedded on the sphere. Any graph G except tree can be embedded on a nonorientable surface. Any graph G can always be embedded on an orientable surface. Let $n_O(G)$ be the number of distinct embeddings on orientable surfaces, then the number of embeddings on all surfaces is

$$2^{\beta(G)} n_O(G),\ n_O(G) = \prod_{i \geq 2} ((i-1)!)^{n_i}, \tag{1.10}$$

where $\beta(G)$ is the Betti number and n_i is the number of vertices of degree i in G.

Proof On a surface of genus not 0, only a graph with at least a circuit is possible to have an embedding. Because a tree has no circuit, it can only embedded on the sphere. Because a graph not a tree has at least one circuit, from Theorem 1.10 the second and the third statements are true. Since distinct planar embeddings of a joint tree of G with the indices of each letter different correspond to distinct embeddings of G on orientable surfaces and the number of distinct planar embeddings of joint trees is

$$n_O(G) = \prod_{i \geq 2} ((i-1)!)^{n_i}.$$

Further, since the indices of letters on the $\beta(G)$ edges has $2^{\beta(G)}$ of choices for a given orientable embedding and among them only one choice corresponds to an orientable embedding, the fourth statement is true. $\square$

For an embedding $\mu(G)$ of G on a surface, let $\nu(\mu G)$, $\epsilon(\mu G)$ and $\phi(\mu G)$ are, respectively, its vertex number, or *order*, edge number, or *size* and face number, or *coorder*.

Theorem 1.12 For a surface S, all embeddings $\mu(G)$ of a graph G have $\mathrm{Eul}(\mu G) = \nu(\mu G) - \epsilon(\mu G) + \phi(\mu G)$ the same, only dependent

on S and independent of G. Further,

$$\mathrm{Eul}(\mu G) = \begin{cases} 2-2p,\ p \geq 0, \\ \quad \text{when } S \text{ has orientable genus } p; \\ 2-q,\ q \geq 1, \\ \quad \text{when } S \text{ has nonorientable genus } q. \end{cases} \tag{1.11}$$

Proof For an embedding $\mu(G)$ on S, if it has at least 2 faces, then by connectedness it has 2 faces with a common edge. From the finite recursion principle, by the inverse of Operation 3 an embedding $\mu(G_1)$ of G_1 on S with only 1 face on S is found. It is easy to check that $\mathrm{Eul}(\mu G)=\mathrm{Eul}(\mu G_1)$. Similarly, by the inverse of Operation 2 an embedding $\mu(G_0)$ of G_0 on S with only 1 vertex is found. It is also easy to check that $\mathrm{Eul}(\mu G)=\mathrm{Eul}(\mu G')$. Further, by Operation 1 and Relations 1–3, it is seen that $\mathrm{Eul}(\mu G_0)=\mathrm{Eul}(O_p)$, $p \geq 0$; or $\mathrm{Eul}(Q_q)$, $q \geq 1$ according as S is an orientable surface in (1.8); or not in (1.9). From the arbitrariness of G, the first statement is proved.

By calculating the order, size and coorder of O_p, $p \geq 0$; or Q_q, $q \geq 1$, (1.11) is soon obtained. So, the second statement is proved. □

According to this theorem, for an embedding $\mu(G)$ of graph G, $\mathrm{Eul}(\mu G)$ is called its *Euler characteristic*, or of the surface it is on. Further, $g(\mu G)$ is the genus of the surface $\mu(G)$ is on.

If a graph G is with the minimum length of circuits σ, then from Theorem 1.12 the genus $\gamma(G)$ of an orientable surface G can be embedded on satisfies the inequality

$$1 - \frac{\nu(G) - \epsilon(1-\frac{2}{\sigma})}{2} \leq \gamma(G) \leq \lfloor \frac{\beta}{2} \rfloor \tag{1.12}$$

and the genus $\tilde{\gamma}(G)$ of a nonorientable surface G can be embedded on satisfies the inequality

$$2 - (\nu(G) - \epsilon(1-\frac{2}{\sigma})) \leq \tilde{\gamma}(G) \leq \beta. \tag{1.13}$$

If a graph has an embedding with its genus attaining the lower(upper) bound in (1.12) and (1.13), then it is called *down*(*up*)-*embeddable*. In

fact, a graph is up-embeddable on nonorientable, or orientable surfaces according as it has an embedding with only 1 face, or at most 2 faces.

Theorem 1.13 All graphs but trees are up-embeddable on nonorientable surfaces.

Further, if a graph has an embedding of nonorientable genus l and an embedding of nonorientable genus k, $l < k$, then for any i, $l < i < k$, it has an embedding of nonorientable genus i.

Proof For an arbitrary embedding of a graph G on a nonorientable surface, let T be its corresponding joint tree. From the nonorientability, the associate $2\beta(G)$-gon P has at least 1 letter with different indices(or same power of its two occurrences!). If $P = Q_q$, $q = \beta(G)$, then the embedding is an up-embedding in its own right. Otherwise, by Relation 2, or Relation 3 if necessary, whenever $s^{-1}s$ or $stst$ occurs, it is, respectively, replaced by ss or $sts^{-1}t$. In virtue of no letter missed in the procedure, from the finite recursion principle, $P' = Q_q$, $q = \beta(G)$, is obtained. This is the first statement.

From the arbitrariness of starting embedding in the procedure of proving the first statement by only using Relation 2 instead of Relation 3 ($AststB \sim_{\text{top}} Ass^{-1}Btt$ by Relation 2), because the genus of the surface is increased 1 by 1, the the second statement is true. □

The second statement of this theorem is also called the *interpolation theorem*. The orientable form of interpolation theorem is firstly given by Duke[Duk1]. The maximum(minimum) of the genus of surfaces (orientable or nonorientable) a graph can be embedded on is call the *maximum genus*(*minimum genus*) of the graph. Theorem 1.13 shows that graphs but trees are all have their maximum genus on nonorientable surfaces the Betti number with the interpolation theorem. The proof would be the simplest one. However, for the orientable case, it is far from simple. many results have been obtained since 1978(see [Liu1–2], [LiuL1], [HuanL1] and [LidL1]) in this aspect. On the determination of minimum genus of a graph, only a few of graphs with certain symmetry are done(see Chapter 12 in [Liu5–6]).

I.4 Abstract representation

Let $G = (V, X)$, $V = \{v_1, v_2, \cdots, v_n\}$,

$$X = \{x_1, x_2, \cdots, x_m\} \subseteq V\{\times\}V = \{\{u, v\}|\forall u, v \in V\},$$

be a graph. For an embedding $\mu(G)$ of G on a surface, each edge has not only two ends as in G but also two sides. Let α be the operation from one side to the other and β be the operation from one end to the other. From the symmetry between the two ends and between the two sides,

$$\alpha^2 = \beta^2 = 1 \tag{1.14}$$

where 1 is the identity. By considering that the result from one side to the other and then to the other end and the result from one end to the other and then to the other side are the same, *i.e.*,

$$\beta\alpha = \alpha\beta. \tag{1.15}$$

Further, it can be seen that $K = \{1, \alpha, \beta, \gamma\}$, $\gamma = \alpha\beta$, is a group, called the *Klein group* where

$$\begin{aligned}(\alpha\beta)^2 &= (\alpha\beta)(\alpha\beta) = (\alpha\beta\alpha)\beta \\ &= (\alpha\alpha\beta)\beta = (\alpha\alpha)(\beta\beta) = 1.\end{aligned} \tag{1.16}$$

Thus, an edge $x \in X$ of G in an embedding $\mu(G)$ of G becomes $Kx = \{x, \alpha x, \beta x, \gamma x\}$, as shown in Fig.1.10.

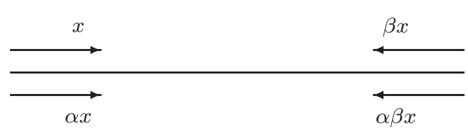


Fig.1.10 An edge sticking on K

In fact, let

$$\mathcal{X} = \sum_{i=1}^{m} Kx_i \tag{1.17}$$

where summation stands for the disjoint union, then α and β can both be seen as a permutation on $\mathcal{X}$, *i.e.*,

$$\alpha = \prod_{i=1}^{m}(x_i, \alpha x_i)(\beta x_i, \gamma x_i), \ \beta = \prod_{i=1}^{m}(x_i, \beta x_i)(\alpha x_i, \gamma x_i).$$

The vertex x is on deals with the rotation as

$$\{(x, \mathcal{P}x, \mathcal{P}^2x, \cdots), (\alpha x, \alpha\mathcal{P}^{-1}x, \alpha\mathcal{P}^{-2}x, \cdots)\}, \tag{1.18}$$

as shown in Fig.1.11(when its degree is 4).

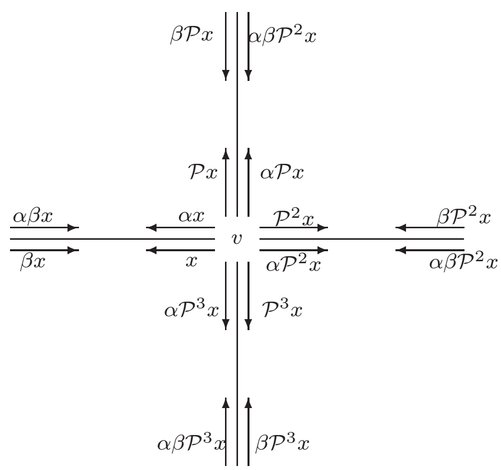


Fig.1.11 The rotation at a vertex

It is seen that $\mathcal{P}$ is also a permutation on $\mathcal{X}$. The set of elements in each cycle of this permutation is called an *orbit* of an element in the cycle. For example, the orbit of element x under permutation $\mathcal{P}$ is denoted by $(x)_\mathcal{P}$. From (1.18),

$$(x)_\mathcal{P} \bigcap (\alpha x)_\mathcal{P} = \emptyset, x \in \mathcal{X}. \tag{1.19}$$

The two cycles at a vertex in an embedding have a relation as

$$\begin{aligned} &(\alpha x, \mathcal{P}\alpha x, \mathcal{P}^2\alpha x, \cdots) \\ =&(\alpha x, \alpha\mathcal{P}^{-1}x, \alpha\mathcal{P}^{-2}x, \cdots) \\ =&\alpha(x, \mathcal{P}^{-1}x, \mathcal{P}^{-2}x, \cdots). \end{aligned} \tag{1.20}$$

For convenience, one of the two cycles is chosen to represent the vertex, *i.e.*,

$$(x, \mathcal{P}x, \mathcal{P}^2x, \cdots),$$

or

$$(\alpha x, \alpha\mathcal{P}^{-1}x, \alpha\mathcal{P}^{-2}x, \cdots).$$

Theorem 1.14 $\alpha\mathcal{P} = \mathcal{P}^{-1}\alpha$.

Proof By multiplying the two sides of (1.20) by α from the left and then comparing the second terms on the two sides,

$$\alpha\mathcal{P}\alpha = \mathcal{P}^{-1}.$$

By multiplying its two sides by α from the right, the theorem is soon obtained. □

Since α and β are both permutations on $\mathcal{X}$, $\gamma = \alpha\beta$ and $\mathcal{P}^* = \mathcal{P}\gamma$ are permutations on $\mathcal{X}$ as well. Let

$$(x, \mathcal{P}^*x, \mathcal{P}^{*2}x, \cdots)$$

be the cycle of $\mathcal{P}^*$ involving x. From the symmetry between βx and x, the cycle of $\mathcal{P}^*$ involving βx is

$$(\beta x, \mathcal{P}^*\beta x, \mathcal{P}^{*2}\beta x, \cdots)$$

which has the same number of elements as that involving x does.

Because $\mathcal{P}^*(\beta x) = \mathcal{P}\alpha\beta(\beta x) = \mathcal{P}\alpha x$ and from Theorem 1.14

$$\begin{aligned}\mathcal{P}\alpha x &= \alpha\mathcal{P}^{-1}x = \alpha\gamma(\gamma\mathcal{P}^{-1})x \\ &= \alpha\gamma\mathcal{P}^{*-1}x \\ &= \beta\mathcal{P}^{*-1}x,\end{aligned}$$

we have

$$\mathcal{P}^*(\beta x) = \beta\mathcal{P}^{*-1}x. \tag{1.21}$$

Furthermore, because $\mathcal{P}^{*2}(\beta x) = \mathcal{P}^*(\mathcal{P}^*(\beta x))$ and from (1.21)

$$\mathcal{P}^*(\mathcal{P}^*(\beta x)) = \mathcal{P}^*(\beta\mathcal{P}^{*-1}x),$$

by (1.21) for $\mathcal{P}^{*-1}x$ instead of x, we have

$$\mathcal{P}^{*2}(\beta x) = \beta(\mathcal{P}^{*-1}(\mathcal{P}^{*-1}x)) = \beta(\mathcal{P}^{*-2}x).$$

On the basis of the finite restrict recursion principle, a cycle is found. Therefore,

$$(\beta x, \mathcal{P}^*\beta x, \mathcal{P}^{*2}\beta x, \cdots) = \beta(x, \mathcal{P}^{*-1}x, \mathcal{P}^{*-2}x, \cdots). \tag{1.22}$$

This implies

$$(x)_{\mathcal{P}^*} \bigcap (\beta x)_{\mathcal{P}^*} = \emptyset, \tag{1.23}$$

for $x \in \mathcal{X}$.

Theorem 1.15 $\beta\mathcal{P}^* = \mathcal{P}^{*-1}\beta$.

Proof A direct result of (1.22). □

Based on (1.22), it is seen that the face involving x of the embedding represented by $\mathcal{P}$ is

$$\{(x, \mathcal{P}^*x, \mathcal{P}^{*2}x, \cdots), (\beta x, \mathcal{P}^{*-1}\beta x, \mathcal{P}^{*-2}\beta x, \cdots)\}. \tag{1.24}$$

Similarly to vertices, based on (1.22) and (1.23), the face can be represented by one of the two cycles in (1.24).

Example 1.3 Let $G = K_4$, *i.e.* , the complete set of order 4. Given its rotation

$$\{(x, y, z), (\beta z, l, \gamma w), (\gamma l, u, \beta y), (\beta x, w, \gamma u)\},$$

as shown in Fig 1.12. Its two faces are $(x, \beta u, \beta l, \gamma z)$ and

$$(y, \alpha u, \alpha w, \alpha l, \gamma y, z, \beta w, \gamma x).$$

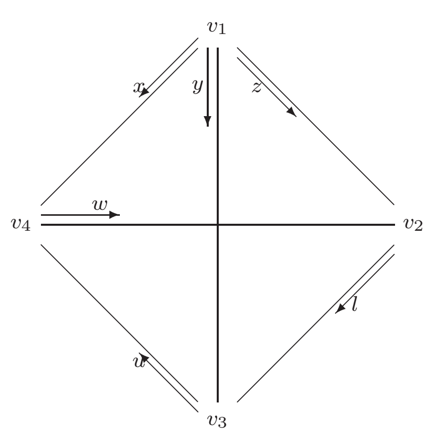


Fig.1.12 A rotation of K_4

Thus, it is an embedding of K_4 on the torus

$$O_1 = (ABA^{-1}B^{-1})$$

as shown in Fig.1.13.

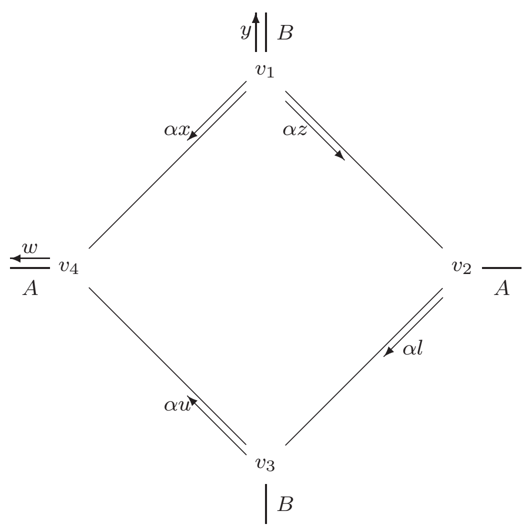


Fig.1.13 Embedding determined by rotation

Further, another rotation of K_4 is chosen for getting another embedding of K_4.

Example 1.4(Continuous to Example 1.3) Another embedding of K_4 is shown as in Fig.1.14. Its rotation is

$$\{(x, y, z), (\beta z, l, \gamma w), (u, \gamma y, \gamma l), (\beta x, w, \gamma u)\}.$$

Its two faces are

$$(x, \beta u, \beta l, \gamma z)$$

and

$$(\alpha x, w, \beta z, \alpha y, \alpha u, \alpha w, \alpha l, \beta y).$$

This is an embedding of K_4 on the Klein bottle

$$N_2 = (ABA^{-1}B) \sim_{\rm top}= (AABB)$$

as shown in Fig.1.14.

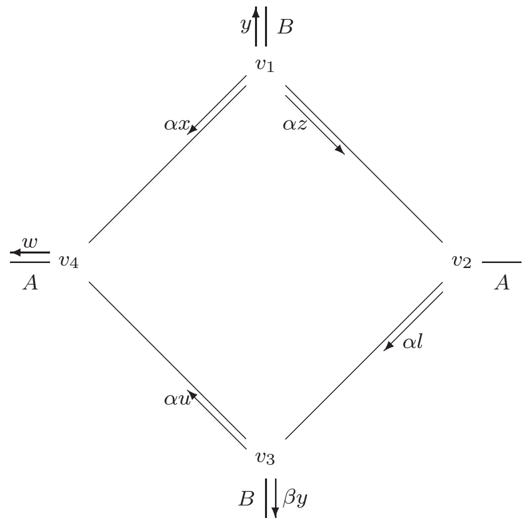


Fig.1.14 Embedding distinguished by rotation

Such an idea is preferable to deal with combinatorial maps via algebraic but neither geometric nor topological approaches.

I.5 Smarandache 2-manifolds with map geometry

Smarandache system The embedding of a graph on surface enables one to construct finitely Smarandache 2-manifolds, i.e., *map geometries* on surfaces.

A rule in a mathematical system $(\Sigma; \mathcal{R})$ is said to be *Smarandachely denied* if it behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways.

A Smarandache system $(\Sigma; \mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule in $\mathcal{R}$ (see [Mao4] for details). Particularly, if $(\Sigma; \mathcal{R})$ is nothing but a metric space $(M; \rho)$, then such a Smarandache system is called a *Smarandache geometry*, seeing references [Mao1]–[Mao4] and [Sma1]–[Sma2].

Example 1.5(Smarandache geometry) Let $\mathbb{R}^2$ be a Euclidean plane, points $A, B \in \mathbf{R}^2$ and l a straight line, where each straight

line passes through A will turn 30^o degree to the upper and passes through B will turn 30^o degree to the down such as those shown in Fig.1.15. Then each line passing through A in F_1 will intersect with l, lines passing through B in F_2 will not intersect with l and there is only one line passing through other points does not intersect with l.

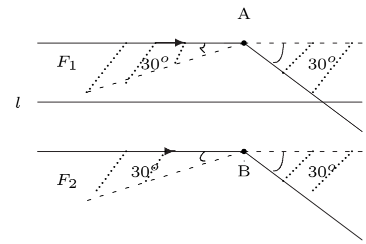


Fig.1.15

Then such a geometry space $\mathbb{R}^2$ with queer points A and B is a Smarandache geometry since the axiom *given a line and a point exterior this line, there is one line parallel to this line* is now replaced by *none line, one line and infinite lines.*

A more general way for constructing Smarandache geometries is by Smarandache multi-spaces ([Mao3]). For an integer $m \geq 2$, let $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ be m mathematical systems different two by two. A *Smarandache multi-space* is a pair $(\widetilde{\Sigma}; \widetilde{\mathcal{R}})$ with

$$\widetilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i, \quad and \quad \widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i.$$

Such a multi-space naturally induce a graph structure with

$$V(G) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$
$$E(G) = \{\ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

Example 1.16([Mao5]) Let n be an integer, $Z_1 = (\{0, 1, 2, \cdots, n-1\}, +)$ an additive group $(mod n)$ and $P = (0, 1, 2, \cdots, n-1)$ a permutation. For any integer $i, 0 \leq i \leq n-1$, define

$$Z_{i+1} = P^i(Z_1)$$

such that $P^i(k) +_i P^i(l) = P^i(m)$ in Z_{i+1} if $k + l = m$ in Z_1, where $+_i$

denotes the binary operation $+_i : (P^i(k), P^i(l)) \to P^i(m)$. Then we know that

$$\left(\bigcup_{i=1}^{n} Z_i; O\right)$$

with $O = \{+_i,\ 0 \le i \le n-1\}$ is a Smarandache multi-space underlying a graph K_n, where $Z_i = Z_1$ for integers $1 \le i \le n$.

Map geometry A nice model on Smarandache geometries, namely *s-manifolds* on the plane was found by Iseri in [Ise1] defined as follows, which is in fact a case of map geometry.

An s-manifold is any collection $\mathcal{C}(T, n)$ of these equilateral triangular disks $T_i, 1 \le i \le n$ satisfying the following conditions:

(i) each edge e is the identification of at most two edges e_i, e_j in two distinct triangular disks $T_i, T_j, 1 \le i, j \le n$ and $i \ne j$;

(ii) each vertex v is the identification of one vertex in each of five, six or seven distinct triangular disks.

These vertices are classified by the number of the disks around them. A vertex around five, six or seven triangular disks is called an *elliptic vertex*, an *Euclidean vertex* or a *hyperbolic vertex*, respectively.

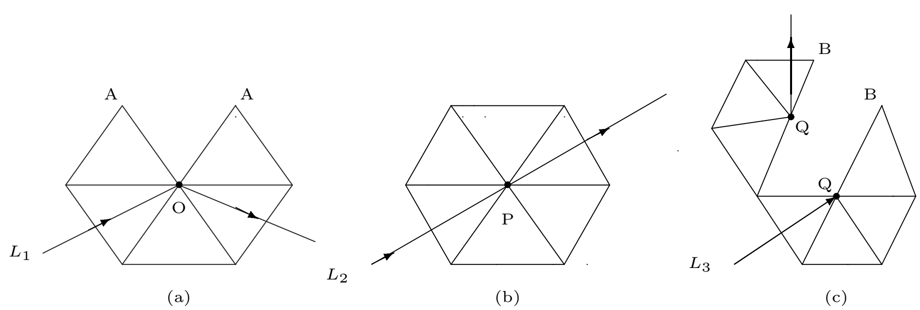


Fig.1.16

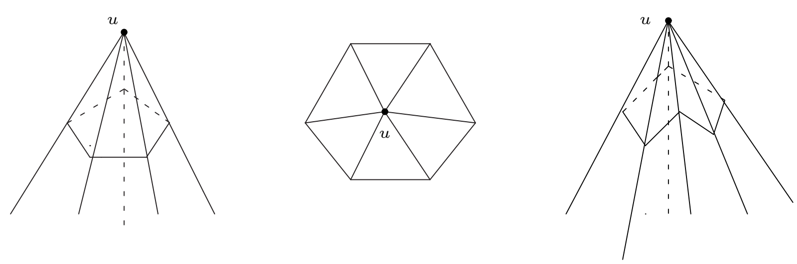
In a plane, an elliptic vertex O, a Euclidean vertex P and a hyperbolic vertex Q and an s-line L_1, L_2 or L_3 passes through points O, P or Q are shown in Fig.1.16$(a), (b), (c)$, respectively.

The *map geometry* is gotten by endowing an angular function $\mu : V(M) \to [0, 4\pi)$ on an embedding M for generalizing Iseri's model

on surfaces following, which was first introduced in [Mao2].

Map geometry without boundary *Let M be a combinatorial map on a surface S with each vertex valency≥ 3 and $\mu : V(M) \to [0, 4\pi)$, i.e., endow each vertex $u, u \in V(M)$ with a real number $\mu(u), 0 < \mu(u) < \frac{4\pi}{\rho_M(u)}$. The pair (M, μ) is called a map geometry without boundary, $\mu(u)$ an angle factor on u and orientable or non-orientable if M is orientable or not.*

Map geometry with boundary *Let (M, μ) be a map geometry without boundary, faces $f_1, f_2, \cdots, f_l \in F(M), 1 \leq l \leq \phi(M) - 1$. If $S(M) \setminus \{f_1, f_2, \cdots, f_l\}$ is connected, then $(M, \mu)^{-l} = (S(M) \setminus \{f_1, f_2, \cdots, f_l\}, \mu)$ is called a map geometry with boundary $f_1, f_2, \cdots, f_l$, and orientable or not if (M, μ) is orientable or not, where $S(M)$ denotes the underlying surface of M.*



$\rho_M(u)\mu(u) < 2\pi$ $\rho_M(u)\mu(u) = 2\pi$ $\rho_M(u)\mu(u) > 2\pi$

Fig.1.17

Certainly, a vertex $u \in V(M)$ with $\rho_M(u)\mu(u) < 2\pi$, $= 2\pi$ or $> 2\pi$ can be also realized in a Euclidean space $\mathbf{R}^3$, such as those shown in Fig.1.17.

A point u in a map geometry (M, μ) is said to be *elliptic*, *Euclidean* or *hyperbolic* if $\rho_M(u)\mu(u) < 2\pi$, $\rho_M(u)\mu(u) = 2\pi$ or $\rho_M(u)\mu(u) > 2\pi$. If $\mu(u) = 60^o$, we find these elliptic, Euclidean or hyperbolic vertices are just the same in Iseri's model, which means that these s-manifolds are a special map geometry. If a line passes through a point u, it must has an angle $\frac{\rho_M(u)\mu(u)}{2}$ with the entering ray and equal to 180^o only when u is Euclidean. For convenience, we always assume that a line passing through an elliptic point turn to the left and a hyperbolic point to the right on the plane. Then we know the following

results.

Theorem 1.16 Let M be an embedding on a locally orientable surface with $|M| \geq 3$ and $\rho_M(u) \geq 3$ for $\forall u \in V(M)$. Then there exists an angle factor $\mu : V(M) \to [0, 4\pi)$ such that (M, μ) is a Smarandache geometry by denial the axiom (E5) with axioms (E5) for Euclidean, (L5) for hyperbolic and (R5) for elliptic.

Theorem 1.17 Let M be an embedding on a locally orientable surface with order≥ 3, vertex valency≥ 3 and a face $f \in F(M)$. Then there is an angle factor $\mu : V(M) \to [0, 4\pi)$ such that $(M, \mu)^{-1}$ is a Smarandache geometry by denial the axiom (E5) with axioms (E5) for Euclidean, (L5) for hyperbolic and (R5) for elliptic.

A complete proof of Theorems 1.16–1.17 can be found in references in [Mao2–4]. It should be noted that the map geometry with boundary is in fact a generalization of *Klein model* for hyperbolic geometry, which uses a boundary circle and lines are straight line segment in this circle, such as those shown in Fig.1.18.

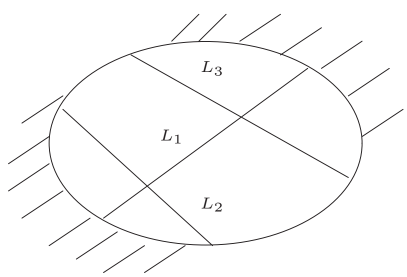


Fig.1.18

Activities on Chapter I

I.6 Observations

O1.1 Let X be a finite set and B be a binary set. Is $\{Bx|x \in X\}$ a pregraph or a graph? If unnecessary, what condition does a pregraph, or a graph satisfy?

O1.2 Let X be a finite set, B be a binary set and

$$\mathcal{X} = \sum_{x \in X} Bx.$$

For a permutation ψ on $\mathcal{X}$ such that $\psi^2 = 1$, is $\{\{x, \psi x\}|x \in X\}$ a graph? If unnecessary, when is it a graph?

O1.3 How many orientable, or nonorientable surfaces can a hexagon represent? List all of them.

O1.4 How many orientable, or nonorientable surfaces can a $2k$-gon represent? How to List them all.

O1.5 In [Liu1], an embedding of a graph G on the nonorientable surface of genus $\beta(G)$ is constructed in a way from a specific tree on G. Now, how to get such an embedding from any tree on G.

O1.6 Suppose G has an embedding on an orientable surface of genus k. If k is not the maximum genus of G, how to find an embedding of G on an orientable surface of genus $k + 1$.

O1.7 Any embedding of a graph $G = (V, X)$ is a permutation on $\{1, \alpha, \beta, \alpha\beta\}X = \mathcal{X}$ as shown by (1.17). However, α, β and $\gamma = \alpha\beta$ are each a permutation on $\mathcal{X}$, but not an embedding of G in general. When does each of them determine an embedding of G.

O1.8 If permutation $\mathcal{P}$ on $\mathcal{X}$ is an embedding of a graph, to show that for any $x \in \mathcal{X}$, there does not exist an integer m such that $\alpha x = \mathcal{P}^m x$.

O1.9 If permutation $\mathcal{P}$ on $\mathcal{X}$ is an embedding of a graph, to show

$$\mathcal{P}\alpha\mathcal{P} = \alpha.$$

O1.10 Observe that permutation β as $\mathcal{P}$ satisfies both O1.8 and O1.9. However, permutation α as $\mathcal{P}$ satisfies O1.9 but not O1.8.

O1.11 Let S be a set of permutations on $\mathcal{X}$ and Ψ_S be the group generated by S. If for any $x, y \in \mathcal{X}$, there exists a $\psi \in \Psi_S$ such that $x = \psi y$, then the group Ψ_S is said to be *transitive* on $\mathcal{X}$. Observe that if permutation $\mathcal{P}$ on $\mathcal{X}$ is an embedding of a graph, then group $\Psi_I, I = \{\mathcal{P}, \alpha, \beta\}$, is transitive on $\mathcal{X}$.

An embedding of a graph $G = (V, E)$ can be combinatorially represented by a rotation system $\sigma(V)$ on the vertex set V of G and a function on the edge set as $\lambda : \ E \longrightarrow B,\ B = \{0, 1\}$, denoted by $G_\sigma(\lambda)$.

In order to determine the faces of an embedding, an immersion of the graph and a rule should be established.

An *immersion* of a graph is such a representation of the graph in the plane that vertices injects into the plane on an imaged circle and edges are straight line segments between their two ends.

Travel and traverse rule From a point on one side of an edge, travel as long as on the same side until at the middle of a edge with $\lambda = 1$, then traverse to the other side.

O1.12 By the TT-rule(*i.e.*, the travel and traverse rule) on an immersion, the initial side the starting point is on can always be encountered to get a travel as a set of edges met on the way.

O1.13 By the TT-rule on an immersion of a graph, a set of travels can always be found for any edge occurs exactly twice.

On a graph $G = (V, E)$, a subset of edges $C \subseteq E$ that V has a 2-*partition* $V = V_1 + V_2$ with the property:

$$C = \{(u, v) \in E | u \in V_1, v \in V_2\} \tag{1.25}$$

is called a *cocycle* of G.

O1.14 For an immersion $G_\sigma(\lambda)$ of graph $G = (V, E)$, the embedding of G determined by $G_\sigma(\lambda)$ is orientable if, and only if, the set $\{e | \forall e \in E, \leq (e) = 1\}$ is a cocycle of G.

I.7 Exercises

E1.1 For a graph G, prove that G has no *odd circuit*(a circuit with odd number of edges) if, and only if, for a tree on G, G has no odd fundamental circuit.

E1.2 Prove that a graph $G = (V, E)$ has no odd fundamental circuit if, and only if, E itself is a cocycle.

Let Γ be a nonAbelian group. The identity is denoted by 1. Write $\Gamma_0 = \{\xi | \xi^2 = 1, \xi \in \Gamma\}$, *i.e.*, the set of all elements of order 2. For a pregraph $G = (V, E)$, let $x_v \in \Gamma_0, v \in V$, be variables on the vertex set V and $w(e) \in \Gamma_0$, $e \in E$ be a weight function on the edge set E. On the network $N = (G; w)$, its *incidence equation* is

$$x_u x_v = w(e), \ e \in E. \tag{1.26}$$

E1.3 Prove that if the incidence equation has a solution, then it has at least $|\Gamma_0|$, *i.e.* , the number of elements in Γ_0, solutions.

E1.4 Prove that the incidence equation has a solution if, and only if, G has no circuit C such that

$$w(C) = \prod_{e \in C} w(e) \neq 1.$$

E1.5 Let $\sigma(G)$ be the number of connected components on pregraph G. Prove that if the incidence equation has a solution, then it

has

$$|\Gamma_0|^{\sigma(G)}$$

solutions.

E1.6 For an orientable surface, provide a procedure for determining its orientable genus and then estimate an upper bound of the operations necessary.

E1.7 For a nonorientable surface, provide a procedure for determining its nonorientable genus and then estimate an upper bound of the number of operations necessarily used.

E1.8 Let G be a graph of order 2 with all its edges selfloops but only one. Prove that G is not up-embeddable on orientable surfaces if, and only if, each vertex is incident with odd number of selfloops.

E1.9 According to the orientable and nonorientable genera, list all embeddings of K_4, the complete graph of order 4.

A graph is called *i-separable* if it has i, $i \geq 1$, vertices such that it is not connected anymore when the i vertices with their incident edges are deleted. A set of i vertices separable without a proper subset separable for a graph is called an *i-cut* of the graph. A graph which has an i-cut without $(i-1)$-cut is said to be *i-connected*.

E1.10 Prove that if a 3-connected graph G is *planar* (embeddable on the sphere), then it has exact 2 embeddings on the sphere.

For a planar graph $G = (V, E)$, and $u, v \in V$, if G has the form as

$$G = \begin{cases} G_1 \bigcup G_2, & \text{当 } (u,v) \in E; \\ G_1 \bigcup G_2 - \{(u,v)\}, & \text{当 } (u,v) \notin E \end{cases} \tag{1.27}$$

and

$$G_1 \bigcap G_2 = (u,v) \tag{1.28}$$

such that G_1 and G_2 are both with at least 2 edges, then $\{u, v\}$ is called a *splitting pair* of G, and G_1 and G_2, its *splitting block*. If a splitting block at a splitting pair has no proper subgraph is still a splitting block, then it is said to be *standard*[Mac1].

E1.11 Prove the following three statements.

(i) Let A and B be two standard splitting blocks of a splitting pair, then they are no edge in common;

(ii) For a splitting pair of a 2-connected planar graph $G=(V,E)$, the *standard splitting block decomposition* of the edge set E, *i.e.*,

$$E = E_1 \bigcup E_2 \bigcup \cdots \bigcup E_s$$

such that $E_i \cap E_j = \emptyset$, $1 \le i \ne j \le s$, and the induced subgraphs of E_i, $1 \le i \le s$, are all standard splitting blocks of the splitting pair, is unique.

(iii) Let b_i be the number of standard splitting blocks of the ith splitting pair, $i = 1, 2, \cdots, m$, m be the number of splitting pairs, then the number of embeddings of a 2-connected planar graph on the sphere is

$$2^{b_1+b_2+\cdots+b_m} \prod_{i=1}^{m} (b_i - 1)!.$$

For a graph G, let $\Pi(G)$ be the set of rotation systems of inner vertices on a joint tree and $\omega_\pi(i)$, $\pi \in \Pi$, be the boundary polygon with its $\beta(G)$ pairs of letters whose indices are determined by a binary number i with $\beta(G)$ digits by the rule: the two indices of the lth letter are same or different according as the lth digit of i is 0 or 1. Define

$$\xi(\omega_\pi(i)) = \sum_{k=-\beta(G)}^{\lfloor \frac{\beta(G)}{2} \rfloor} a_k x^k \tag{1.29}$$

where x is an undeterminate and

$$a_k = \begin{cases} 1, & \text{if } \omega_\pi(i) \sim_{\text{top}} O_k, \text{ or } Q_{-k}; \\ 0, & \text{otherwise.} \end{cases} \tag{1.30}$$

Now, let

$$\Omega_\pi = \{\omega_\pi(i) |\ 0 \le i \le 2^{\beta(G)} - 1\} \tag{1.31}$$

and

$$\xi(\Omega_\pi) = \sum_{i=0}^{2^{\beta(G)}-1} \xi(\omega_\pi(i)). \tag{1.32}$$

E1.12 Prove that the coefficient of x^k in

$$\xi(G) = \sum_{\pi\in\Pi(G)} \xi(\Omega_\pi)$$

is the number of embeddings of G on the surface of relative genus k, $-\beta(G) \le k \le \lfloor\frac{\beta(G)}{2}\rfloor$.

A graph of order 2 without selfloop is called a *link bundle.*

E1.13 Let L_m be the link bundle of size $m \ge 1$. Determine $\xi(L_m)$.

A graph of order 1 is also called a *bouquet*, or a *loop bundle.*

E1.14 Let B_m be a bouquet of size m, $m \ge 1$. Determine $\xi(B_m)$.

A graph of order 2 is called a *bipole.* Of course, a link bundle is a bipole which has no selfloop.

E1.15 Let P_m be a bipole of size m, $m \ge 1$. Determine $\xi(P_m)$.

I.8 Researches

The set(repetition at most twice of elements permitted) of edges appearing on a travel can be shown to have a partition of each subset forming still a subtravel except probably the travel itself. Such a partition is called a *decomposition* of a travel into subtravels. However, it is not yet known if any travel can be decomposed into tours except only the case that its induced graph has a cut-edge.

R1.1 Prove, or disprove, the conjecture that a travel with at most twice occurrences of an edge in a graph has a decomposition into tours if, and only if, the induced subgraph of the travel is without cut-edge.

Because a circuit is restricted from a tour by no repetition of a vertex, the following conjecture would look stronger the last one.

R1.2 Prove, or disprove, the conjecture that a travel with at

most twice occurrences of an edge in a graph has a decomposition into circuits if, and only if, the induced subgraph of the travel is without cut-edge.

However, it can be shown from Theorem 1.3 that any tour has a decomposition into circuits. The above two conjectures are, in fact, equivalent. Because a cut-edge is never on a circuit, the necessity is always true. A travel with three occurrences of an edge permitted does not have a decomposition into circuits in general. For example, on the graph determined by Par$=\{\{x(0), y(0)\}, \{x(1), y(1)\}\}$, the travel $xx^{-1}xy^{-1}$ where $x = \langle x(0), x(1)\rangle$ and $y = \langle y(0), y(1)\rangle$ has no circuit decomposition.

Furthermore, the two conjectures are apparently right when the graph is planar because each face boundary of its planar embedding is generally a tour whenever without cut-edge.

R1.3 For a given graph G and an integer p, $p \geq 0$, find the number $n_p(G)$ of embeddings of G on the orientable surface of genus p.

The aim is at the genus distribution of embeddings of G on orientable surfaces, *i.e.*, the polynomial

$$P_O(G) = \sum_{p=0}^{\lfloor \sigma/2 \rfloor} n_p(G)x^p,$$

where σ is the Betti number of G.

For $p = 0$, $n_0(G)$ can be done based on [Liu6]. If G is planar, O1.11 provides the result for 2-connected case. Others can also be derived. As to justify if a graph is planar, a theory can be seen from Chapters 3,5 and 7 in [Liu5].

Generally speaking, not easy to get the complete answer in a short period of time. However, the following approach would be available to access this problem. Choose a special type of graphs, for instance, a *wheel*(a circuit C_n all of whose vertices are adjacent to an extra vertex), a *generalized Halin graph*(a circuit with a disjoint tree

except for all articulate vertices forming the vertex set of the circuit) and so forth.

Of course, the technique and theoretical results in 1.3 can be employed to calculate the number of distinct embeddings of a graph by hand and by computer.

R1.4 *Orientable single peak conjecture.* The coefficients of the polynomial in R1.3 are of *single peak* , *i.e.*, they are from increase to decrease as p runs from 0 to $\lfloor \sigma/2 \rfloor (\geq 3)$, $n_p(G)$.

The purpose here is to prove, or disprove the conjecture not necessary to get all $n_p(G)$, $0 \leq p \leq \lfloor \sigma/2 \rfloor (\geq 3)$.

R1.5 Determine the number of distinct embeddings, which have one, or two faces, of a graph on orientable surfaces.

R1.6 For a given graph G and an integer q, $q \geq 1$, find the number $\tilde{n}_q(G)$ of distinct embeddings on nonorientable surfaces of genus q.

The aim is at the genus polynomial of embeddings of G on nonorientable surfaces:

$$P_N(G) = \sum_{q=1}^{\sigma} \tilde{n}_q(G) x^q,$$

where σ is the Betti number of G.

Some pre-investigations for G is that a wheel, or a generalized Halin graph can firstly be done.

R1.7 For a graph G, justify if it is embeddable on the projective plane, and then determine $\tilde{n}_1(G)$ according to the connectivity of G.

R1.8 For a graph embeddable on the projective plane, determine how many sets of circuits such that for each, all of its circuits are essential if, and only if, one of them is essential in an embedding of G on the projective plane.

R1.9 *Nonorientable single peak conjecture.* The coefficients of the polynomial in R1.6 are of single peak in the interval $[o, \sigma]$ where σ is the Betti number of G.

R1.10 For a given type of graphs $\mathcal{G}$ and an integer p, find the number of distinct embeddings of graphs in $\mathcal{G}$ on the orientable surface of genus p. Further, determine the polynomial

$$P_O(\mathcal{G}) = \sum_{p=0}^{\lfloor \sigma(\mathcal{G})/2 \rfloor} n_p(\mathcal{G}) x^p$$

where $\sigma(\mathcal{G}) = \max\{\sigma(G) | G \in \mathcal{G}\}$.

R1.11 For a given type of graphs $\mathcal{G}$ and an integer q, $q \geq 1$, find the number of embeddings of graphs in $\mathcal{G}$ on the nonorientable surface of genus q. Further, determine the polynomial

$$P_N(\mathcal{G}) = \sum_{q=1}^{\sigma(\mathcal{G})} \tilde{n}_q(\mathcal{G}) x^q$$

where $\sigma(\mathcal{G}) = \max\{\sigma(G) | G \in \mathcal{G}\}$.

R1.12 For a set of graphs with some fixed invariants, extract sharp bounds(lower or upper) of the orientable minimum genus and sharp bounds(lower or upper) of orientable maximum genus.

Here, invariants are chosen from the *order* (vertex number), *size*(edge number), *chromatic number* (the minimum number of colors by which vertices of a graph can be colored such that adjacent vertices have distinct colors), *crossing number*(the minimum number of crossing inner points among all planar immersions of a graph), *thickness* (the minimum number of subsets among all partitions of the edge set such that each of the subsets induces a planar graph), and so forth.

R1.13 For a set of graphs and a set of invariants fixed, provide sharp bounds(lower or upper) of minimum nonorientable genus of embeddings of graphs in the set.

Chapter II

Abstract Maps

- A ground set is formed by the Klein group $K = \{1, \alpha, \beta, \alpha\beta\}$ sticking on a finite set X.
- A basic permutation is such a permutation on the ground set that no element x is in the same cycle with αx.
- The conjugate axiom on a map is determined by each vertex consisting of two conjugate cycles for α, as well as by each face consisting of two conjugate cycles for β.
- The transitive axiom on a map is from the connectedness of its underlying graph.
- An included angle is determined by either a vertex with one of its incident faces, or a face with one of its incident vertices.

II.1 Ground sets

Given a finite set $X = \{x_1, x_2, \cdots, x_m\}$, called the *basic set*, its elements are distinct. Two operations α and β on X are defined as for any $x \in X$, $\alpha x \neq \beta x$, $\alpha x, \beta x \notin X$ and $\alpha^2 x = \alpha(\alpha x) = x$, $\beta^2 x = \beta(\beta x) = x$. Further, define $\alpha\beta = \beta\alpha = \gamma$ such that for any $x \in X$, $\gamma x \neq \alpha x, \beta x$ and $\gamma x \notin X$.

Let $\alpha X = \{\alpha x | \forall x \in X\}$, $\beta X = \{\beta x | \forall x \in X\}$ and $\gamma X = \{\gamma x | \forall x \in X\}$, then α, β and γ determine a *bijection* between X and, respectively, αX, βX and γX. By a bijection is meant a one to one correspondence between two sets of the same cardinality. In other words,

$$\begin{aligned} & X \bigcap \alpha X = X \bigcap \beta X = X \bigcap \gamma X = \emptyset; \\ & \alpha X \bigcap \beta X = \beta X \bigcap \gamma X = \gamma X \bigcap \alpha X = \emptyset; \\ & |\alpha X| = |\beta X| = |\gamma X| = |X|. \end{aligned} \tag{2.1}$$

The set $X \cup \alpha X \cup \beta X \cup \gamma X$, or briefly $\mathcal{X} = \mathcal{X}(X)$, is called the *ground set*.

Now, observe set $K = \{1, \alpha, \beta, \gamma\}$. Its elements are seen as permutations on the ground set $\mathcal{X}$. Here, 1 is the identity. From $\alpha^2 = \beta^2 = 1$ and $\alpha\beta = \beta\alpha$, $\gamma^2 = (\alpha\beta)(\alpha\beta) = \alpha(\beta\beta)\alpha = \alpha^2 = 1$, and hence

$$\begin{aligned} \alpha &= \prod_{i=1}^{m} (x_i, \alpha x_i) \prod_{i=1}^{m} (\beta x_i, \gamma x_i); \\ \beta &= \prod_{i=1}^{m} (x_i, \beta x_i) \prod_{i=1}^{m} (\alpha x_i, \gamma x_i); \\ \gamma &= \prod_{i=1}^{m} (x_i, \gamma x_i) \prod_{i=1}^{m} (\beta x_i, \alpha x_i). \end{aligned} \tag{2.2}$$

It is easily seen that K is a group, called *Klein group* because it is isomorphic to the group of four elements discovered by Klein in geometry.

For any $x \in \mathcal{X}$, let $Kx = \{x, \alpha x, \beta x, \gamma x\}$, called a *quadricell*.

Theorem 2.1 For any basic set X, its ground set is

$$\mathcal{X} = \sum_{x \in X} Kx, \tag{2.3}$$

where the summation represents the disjoint union of sets.

Proof From (2.1), for any $x, y \in X$ and $x \neq y$,

$$Kx \bigcap Ky = \{x, \alpha x \beta x, \gamma x\} \bigcap \{y, \alpha y, \beta y, \gamma y\} = \emptyset.$$

From (2.1) again,

$$\bigcup_{x\in X} Kx = X + \alpha X + \beta X + \gamma X = \mathcal{X}.$$

Therefore, (2.3) is true. $\square$

Furthermore, since for any $x \in X$,

$$K(\alpha x) = K(\beta x) = K(\gamma x) = Kx, \tag{2.4}$$

we have

$$\mathcal{X} = \sum_{y\in\alpha X} Ky = \sum_{z\in\beta X} Kz = \sum_{t\in\gamma X} Kt.$$

This implies that the four elements in a quadricell are with symmetry.

II.2 Basic permutations

Let $\mathrm{Per}(\mathcal{X})$, or briefly Per, be a permutation on the ground set $\mathcal{X}$. Because of bijection, according to the finite strict recursion principle, for any $x \in \mathcal{X}$, there exists a minimum positive integer $k(x)$ such that

$$\mathrm{Per}^{k(x)}x = x,$$

i.e., Per contains the *cyclic permutation* , or in short *cycle*,

$$(x)_{\mathrm{Per}} = (x, \mathrm{Per}^2 x, \cdots, \mathrm{Per}^{k(x)-1}x).$$

Write $\{x\}_{\mathrm{Per}}$ as the set of all elements in the cycle $(x)_{\mathrm{Per}}$. Such a set is called the *orbit* of x under permutation Per. The integer $k(x)$ is called the *order* of x under permutation Per.

If for any $x \in \mathcal{X}$,

$$\alpha x \notin \{x\}_{\mathrm{Per}}, \tag{2.5}$$

then the permutation Per is said to be *basic* to α.

Example 2.1 From (2.2), for permutations α and β on the ground set, α is not basic, but β is basic.

Let Par$=\{X_1, X_2, \cdots, X_s\}$ be a partition on the ground set $\mathcal{X}$, then

$$\mathrm{Per} = \prod_{i=1}^{s}(X_i) \tag{2.6}$$

determines a permutation on $\mathcal{X}$, called *induced* from the partition Par. Here, $(X_i),\ 1 \leq i \leq s$, stands for a cyclic order arranged on X_i. This shows that a partition $\{X_i|1 \leq i \leq s\}$ has

$$\prod_{s=1}^{s}(|X_i| - 1)! = \prod_{i\geq 3}((i-1)!)^{n_i} \tag{2.7}$$

induced permutations. In (2.7), $n_i,\ i \geq 3$, is the number of subsets in Par with i elements.

Theorem 2.2 Let Par$=\{X_i|1 \leq i \leq s\}$ be a partition on the ground set $\mathcal{X}(X)$. If Par has an induced permutation basic, then all of its induced permutations are basic. Further, a partition Par has its induced permutation basic if, and only if, for $x \in \mathcal{X}$, there does not exist $Y \in$ Par such that

$$\{x, \alpha x\} \text{ or } \{\beta x, \gamma x\} \subseteq Y \bigcap Kx. \tag{2.8}$$

Proof Because the basicness of a permutation is independent of the order on cycles, the first statement is proved.

Assume an induced permutation Per of a partition Par is basic. From (2.5), for any $x \in X$, in virtue of

$$Kx = \{x, \alpha x\} + \{\beta x, \gamma x\},$$

no $Y\in$Par exists such that (2.8) is satisfied. This is the necessity of the second statement.

Conversely, because for any $x \in \mathcal{X}$, no $Y \in$ Par exists such that (2.8) is satisfied, it is only possible that x and αx are in distinct subsets of partition Par. Therefore, $\alpha x \notin \{x\}_{\mathrm{Per}}$. Based on (2.5), this is the sufficiency of the second statement. □

On the basis of this theorem, induced basic permutations can be easily extracted from a partition of the ground set.

Example 2.2 Let $\mathcal{X} = \{x, \alpha x, \beta x, \gamma x\} = Kx$. There are 15 partitions on $\mathcal{X}$ as

$$\begin{aligned}
&\mathrm{Par}_1 = \{\{x\}, \{\alpha x\}, \{\beta x\}, \{\gamma x\}\};\\
&\mathrm{Par}_2 = \{\{x, \alpha x\}, \{\beta x\}, \{\gamma x\}\};\\
&\mathrm{Par}_3 = \{\{x, \beta x\}, \{\alpha x\}, \{\gamma x\}\};\\
&\mathrm{Par}_4 = \{\{x, \gamma x\}, \{\alpha x\}, \{\beta x\}\};\\
&\mathrm{Par}_5 = \{\{\beta x, \alpha x\}, \{x\}, \{\gamma x\}\};\\
&\mathrm{Par}_6 = \{\{x\}, \{\alpha x, \gamma x\}, \{\beta x\}\};\\
&\mathrm{Par}_7 = \{\{x\}, \{\alpha x\}, \{\beta x, \gamma x\}\};\\
&\mathrm{Par}_8 = \{\{x, \alpha x, \beta x\}, \{\gamma x\}\};\\
&\mathrm{Par}_9 = \{\{x, \beta x, \gamma x\}, \{\beta x\}\};\\
&\mathrm{Par}_{10} = \{\{x, \alpha x, \gamma x\}, \{\alpha x\}\};\\
&\mathrm{Par}_{11} = \{\{x\}, \{\alpha x, \beta x, \gamma x\}\};\\
&\mathrm{Par}_{12} = \{\{x, \alpha x, \beta x, \gamma x\}\};\\
&\mathrm{Par}_{13} = \{\{x, \alpha x\}, \{\beta x, \gamma x\}\};\\
&\mathrm{Par}_{14} = \{\{x, \beta x\}, \{\alpha x, \gamma x\}\};\\
&\mathrm{Par}_{15} = \{\{x, \gamma x\}, \{\alpha x, \beta x\}\}.
\end{aligned}$$

From Theorem 2.2, induced basic permutations can only be extracted from Par_1, Par_3, Par_4, Par_5, Par_6, Par_{14} and Par_{15} among them. Since each of these partitions has no subset with at least 3 elements, from (2.7) it only induces 1 basic permutation. Hence, 7 basic permutations are induced in all.

Based on Theorem 2.2, a partition that induces a basic permutation is called *basic* as well.

For a partition Par on $\mathcal{X}$, if every pair of x and αx, $x \in \mathcal{X}$, deals with the element x in Par, then this partition determines a pregraph if any. For example, in Example 2, there are only

$$\begin{aligned}
&\mathrm{Par}_1 = \mathrm{Par}_2 = \mathrm{Par}_7 = \mathrm{Par}_{13} = \{\{x\}, \{\beta x\}\};\\
&\mathrm{Par}_{12} = \mathrm{Par}_{14} = \mathrm{Par}_{15} = \{\{x, \beta x\}\}
\end{aligned}$$

form 2 premaps of size 1 and others meaningless among the 15 par-

titions. Further, each of the 2 premaps is a graph. The result is the same as in Example 1.1.

II.3 Conjugate axiom

Let Per_1 and Per_2 be two permutations on the ground set $\mathcal{X}$. If for $x \in \mathcal{X}$,

$$(\mathrm{Per}_2 x)_{\mathrm{Per}_1} = \mathrm{Per}_2(x)_{\mathrm{Per}_1^{-1}} = \mathrm{Per}_2(x)^{-1}_{\mathrm{Per}_1}, \tag{2.9}$$

then the two orbits $(x)_{\mathrm{Per}_1}$ and $(\mathrm{Per}_2 x)_{\mathrm{Per}_1}$ of Per_1 are said to be *conjugate*.

For a permutation $\mathcal{P}$ on the ground set $\mathcal{X}$, if

$$\alpha\mathcal{P} = \mathcal{P}^{-1}\alpha, \tag{2.10}$$

then $(\mathcal{P}, \alpha)$(or for the sake of brevity, $\mathcal{P}$) is called satisfying the *conjugate axiom*.

Theorem 2.3 For a basic permutation $\mathcal{P}$ on the ground set $\mathcal{X}$, the two orbits $(x)_\mathcal{P}$ and $(\alpha x)_\mathcal{P}$ for any $x \in \mathcal{X}$ are conjugate if, and only if, $(\mathcal{P}, \alpha)$ satisfies the conjugate axiom.

Proof Necessity. Because of orbits $(x)_\mathcal{P}$ and $(\alpha x)_\mathcal{P}$ conjugate, from (2.9), $(\alpha x)_\mathcal{P} = \alpha(x)^{-1}_\mathcal{P}$. Hence, $\mathcal{P}\alpha x = \alpha\mathcal{P}^{-1}x$, *i.e.*, $\mathcal{P}\alpha = \alpha\mathcal{P}^{-1}$. This implies (2.10).

Sufficiency. Since $\mathcal{P}$ satisfies (2.10),

$$\begin{aligned}\mathcal{P}(\alpha x) &= \mathcal{P}(\alpha\mathcal{P})\mathcal{P}^{-1}x = \mathcal{P}(\mathcal{P}^{-1}\alpha)\mathcal{P}^{-1}x \\ &= (\mathcal{P}\mathcal{P}^{-1})\alpha\mathcal{P}^{-1}x = \alpha\mathcal{P}^{-1}x.\end{aligned}$$

By induction, assume that $\mathcal{P}^l(\alpha x) = \alpha\mathcal{P}^{-l}x$, $l \geq 1$, then we have

$$\begin{aligned}\mathcal{P}^{l+1}(\alpha x) &= \mathcal{P}\mathcal{P}^l(\alpha x) = \mathcal{P}\mathcal{P}^{-l}x \\ &= \mathcal{P}(\alpha\mathcal{P}\mathcal{P}^{-1})\mathcal{P}^{-l}x \text{ (then by (2.10))} \\ &= (\mathcal{P}\mathcal{P}^{-1})\alpha\mathcal{P}^{-(l+1)}x \\ &= \alpha\mathcal{P}^{-(l+1)}x.\end{aligned}$$

Hence, $(\alpha x)_{\mathcal{P}} = \alpha(x)_{\mathcal{P}}^{-1}$. From (2.9), orbits $(x)_{\mathcal{P}}$ and $(\alpha x)_{\mathcal{P}}$ are conjugate. This is the sufficiency. $\square$

Unlike Theorem 2.2, for a partition on the ground set $\mathcal{X}$, from one of its induced permutations satisfying the conjugate axiom, it can not be deduced to others.

A *premap*, denoted by $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, is such a basic permutation $\mathcal{P}$ on the ground set $\mathcal{X}$ that the conjugate axiom is satisfied for $(\mathcal{P}, \alpha)$.

Example 2.3 By no means any basic partition is in companion with a basic permutation. Among the 7 basic partitions as shown in Example 2.2, only the induced permutations of Par_1, Par_{14} and Par_{15} are premaps.

Because $(\mathcal{P}, \beta)$ is not necessary to satisfy the conjugate axiom on a premap $(\mathcal{X}, \mathcal{P})$, α is called the *first operation* and β, the *second* . Thus, $\mathcal{X}$ should be precisely written as $\mathcal{X}_{\alpha,\beta}$ if necessary.

Based on the basicness and Theorem 2.3, any premap $\mathcal{P}$ has the form as

$$\prod_{x \in X_{\mathcal{P}}} (x)_{\mathcal{P}} (x)_{\mathcal{P}}^{*}, \tag{2.11}$$

where $X_{\mathcal{P}}$ is the set of distinct representatives for the conjugate pairs $\{\{x\}_{\mathcal{P}}, \{x\}_{\mathcal{P}}^{*}\}$ of cycles in $\mathcal{P}$. And further,

$$\mathcal{X} = \sum_{x \in X_{\mathcal{P}}} \{x\}_{\mathcal{P}} \{x\}_{\mathcal{P}}^{*}. \tag{2.12}$$

For convenience, one of two conjugate orbits in $\{\{x\}_{\mathcal{P}}, \{x\}_{\mathcal{P}}^{*}\}$ is chosen to stand for the pair itself as a *vertex* of the premap.

Example 2.4 Let $X = \{x_1, x_2\}$, then

$$\mathcal{X} = \{x_1, \alpha x_1, \beta x_1, \gamma x_1, x_2, \alpha x_2, \beta x_2, \gamma x_2\}.$$

Choose

$$\mathcal{P}_1 = (x_1, \beta x_1)(\alpha x_1, \gamma x_1)(x_2)(\alpha x_2)(\beta x_2)(\gamma x_2)$$

and

$$\mathcal{P}_2 = (x_1, \beta x_1, x_2)(\alpha x_1, \alpha x_2, \gamma x_1)(\beta x_2)(\gamma x_2).$$

The former has 3 vertices $(x_1, \beta x_1)$, (x_2) and (βx_2). The latter has 2 vertices $(x_1, \beta x_1, x_2)$ and (βx_2).

Lemma 2.1 If permutation $\mathcal{P}$ on $\mathcal{X}_{\alpha,\beta}$ is a premap, then

$$\mathcal{P}^*\beta = \beta\mathcal{P}^{*-1}, \tag{2.13}$$

where $\mathcal{P}^* = \mathcal{P}\gamma$, $\gamma = \alpha\beta$.

Proof Because $\mathcal{P}^*\beta = \mathcal{P}\alpha\beta\beta = \mathcal{P}\alpha$, from the conjugate axiom,

$$\begin{aligned}\mathcal{P}^*\beta &= \alpha\mathcal{P}^{-1} \\ &= \beta\beta\alpha\mathcal{P}^{-1} \quad (\text{因 } \beta^2 = 1) \\ &= \beta((\mathcal{P}\alpha\beta)^{-1}) \\ &= \beta\mathcal{P}^{*-1}.\end{aligned}$$

Therefore, the lemma holds. □

This lemma tells us that although β does not satisfy the conjugate axiom for permutation $\mathcal{P}$ in general, β does satisfy the conjugate axiom for permutation $\mathcal{P}^*$.

Lemma 2.2 If permutation $\mathcal{P}$ on $\mathcal{X}_{\alpha,\beta}$ is a premap, then permutation $\mathcal{P}^* = \mathcal{P}\gamma$ is basic for β.

Proof Because the 4 elements in a quadricell are distinct, $x \neq \beta x$.

Case 1 $\mathcal{P}^*x \neq \beta x$. Otherwise, from $\mathcal{P}^*x = \beta x$, $\mathcal{P}\alpha(\beta x) = \mathcal{P}^*x = \beta x$. A contradiction to that $\mathcal{P}$ is basic for α.

Case 2 $(\mathcal{P}^*)^2x \neq \beta x$. Otherwise, $\mathcal{P}^*x = \mathcal{P}^{*-1}\beta x$. From Lemma 2.1, $\mathcal{P}^*x = \beta\mathcal{P}^*x$. A contradiction to that $\mathcal{P}^*x$ and $\beta\mathcal{P}^*x$ are in the same quadricell.

In general, assume by induction that Case l: $(\mathcal{P}^*)^l x \neq \beta x$, $1 \leq l \leq k$, $k \geq 2$, is proved. To prove

Case $k+1$ $(\mathcal{P}^*)^{k+1}x \neq \beta x$. Otherwise, $\mathcal{P}^{*k}x = \mathcal{P}^{*-1}\beta x$. From Lemma 2.1, $\mathcal{P}^{*k-1}(\mathcal{P}^*x) = \beta(\mathcal{P}^*x)$. A contradiction to the induction hypothesis.

In all, the lemma is proved. □

Theorem 2.4 Permutation $\mathcal{P}$ on $\mathcal{X}$ is a premap $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ if, and only if, permutation $\mathcal{P}^* = \mathcal{P}\gamma$ is a premap $(\mathcal{X}_{\beta,\alpha}, \mathcal{P}^*)$.

Proof Necessity. Since permutation $\mathcal{P}$ is a premap $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, $\mathcal{P}$ is basic for α and satisfies the conjugate axiom. From Lemma 2.2 and Lemma 2.1, $\mathcal{P}^*$ is basic for β and satisfies the conjugate axiom. Hence, $\mathcal{P}^*$ is a premap $(\mathcal{X}_{\beta,\alpha}, \mathcal{P}^*)$.

Sufficiency. Because $\mathcal{P}^{**} = \mathcal{P}^*\gamma = \mathcal{P}$, the sufficiency is right. □

On the basis of Theorem 2.4, the vertices of premap $\mathcal{P}^*$ are defined to be the *faces* of premap $\mathcal{P}$. The former is called the *dual* of the latter. Since $\mathcal{P}^{**} = \mathcal{P}$, the latter is also the dual of the former.

Example 2.5 For the two premaps $\mathcal{P}_1$ and $\mathcal{P}_2$ as shown in Example 2.4, we have

$$\mathcal{P}_1^* = (x_1, \alpha x_1)(\beta x_1, \gamma x_1)(x_2, \gamma x_2)(\beta x_2, \alpha x_2)$$

and

$$\mathcal{P}_2^* = (x_1, \alpha x_1, x_2, \gamma x_2)(\beta x_1, \alpha x_2, \beta x_2, \gamma x_1).$$

Because $\mathcal{P}_1^*$ has 2 vertices $(x_1, \alpha x_1)$ and $(x_2, \gamma x_2)$ and $\mathcal{P}_2^*$ has 1 vertex $(x_1, \alpha x_1, x_2, \gamma x_2)$, we seen that premaps $\mathcal{P}_1$ and $\mathcal{P}_2$ have, respectively, 2 faces and 1 face.

II.4 Transitive axiom

For a set of permutations $T = \{\tau_i | 1 \leq i \leq k\}$, $k \geq 1$ on $\mathcal{X}$, let

$$\Psi_T = \{\psi | \psi = \prod_{l=1}^{s} \prod_{j=1}^{k} \tau_{\pi_l(j)}^{i_j(\pi_l)},\ i_j(\pi_l) \in Z, \pi_l \in \Pi, s \geq 1\}, \tag{2.14}$$

where Z is the set of integers and Π is the set of all permutations on $\{1, 2, \cdots, k\}$, $k \geq 1$.

Since all elements in Ψ_T are permutations on $\mathcal{X}$, they are closed for composition(or in other word, multiplication) with the associative law but without the commutative law.

Further, it is seen that a permutation in Ψ_T if, and only if, its inverse is in Ψ. The identity is the element in Ψ_T when all $i_j(\pi) = 0$, $\pi \in \Pi$ in (2.14). Therefore, Ψ_T is a group in its own right, called the *generated group* of T.

Let Ψ be a permutation group on $\mathcal{X}$. If for any $x, y \in \mathcal{X}$, there exists an element $\psi \in \Psi$ such that $x = \psi y$, then the group Ψ is said to be *transitive* , or in other words, the group Ψ satisfies the *transitive axiom*.

Now, consider a binary relation on $\mathcal{X}$, denoted by $\sim_\Psi$, that for any $x, y \in \mathcal{X}$,

$$x \sim_\Psi y \iff \exists\ \psi \in \Psi,\ x = \psi y. \tag{2.15}$$

Because the relation $\sim_\Psi$ determined by (2.15) for a permutation group Ψ on $\mathcal{X}$ is a equivalence, $\mathcal{X}$ is classified into classes as $\mathcal{X}/\sim_\Psi$.

Theorem 2.5 A permutation group Ψ on $\mathcal{X}$ is transitive if, and only if, for the equivalence $\sim_\Psi$ determined by (2.15), $|\mathcal{X}/\sim_\Psi| = 1$.

Proof Necessity. From the transitivity, for any $x, y \in \mathcal{X}$, there exists $\psi \in \Psi$ such that $x = \psi y$. In view of (2.15), for any $x, y \in \mathcal{X}$, $x \sim_\Psi y$. Hence, for $\sim_\Psi$, $|\mathcal{X}/\sim_\Psi| = 1$.

Sufficiency. Because $|\mathcal{X}/\sim_\Psi| = 1$, for any $x, y \in \mathcal{X}$, exists $\psi \in \Psi$ such that $x = \psi y$. Therefore, the permutation group Ψ on $\mathcal{X}$ is transitive. $\square$

For a premap, the pregraph with the same vertices and edges as the premap is is called its *under pregraph*. Conversely, the premap is a *super premap* of its under pregraph.

Let $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ be a premap. If permutation group Ψ_J, $J = \{\alpha, \beta, \mathcal{P}\}$, on the ground set $\mathcal{X}_{\alpha,\beta}$ is transitive, *i.e.*, with the *transitive axiom*, then the premap is called a *map* .

Lemma 2.3 Let $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ be a premap. For any $x, y \in$

$\mathcal{X}_{\alpha,\beta}$, exists $\psi \in \Psi_J$, $J = \{\alpha, \beta, \mathcal{P}\}$, such that $x = \psi y$ if, and only if, there is a path from the vertex v_x x is with to the vertex v_y y is with in the under pregraph of M.

Proof Necessity. In view of (2.14) with the conjugate axiom, write

$$\psi = \alpha^{\delta_1}\mathcal{P}^{l_1}\alpha^{\sigma_1}\beta\alpha^{\delta_2}\mathcal{P}^{l_2}\alpha^{\sigma_2}\beta\cdots\beta\alpha^{\delta_s}\mathcal{P}^{l_s}\alpha^{\sigma_s},$$

where $\delta_i, \sigma_i \in \{0,1\}$, $1 \le i \le s$, and $l_i \in Z$, $1 \le i \le s$. Because $\alpha^{\delta_s}\mathcal{P}^{l_s}\alpha^{\sigma_s}$ and $\alpha^{\delta_1}\mathcal{P}^{l_1}\alpha^{\sigma_1}$ are, respectively, acting on vertices v_y and v_x, ψ determines a trail from v_y to v_x of $s-1$ edges. Since there is a trail between two vertices if, and only if, there is a path between them, the necessity is done.

Sufficiency. let $\langle v_s, v_{s-1}, \cdots, v_1\rangle$,$v_s = v_y$, $v_1 = v_x$, be a path from v_y to v_x in the under pregraph of M. Then, there exist $\delta_i, \sigma_i \in \{0,1\}$, $1 \le i \le s$, and $l_i \in Z$, $1 \le i \le s$, such that

$$\psi = \alpha^{\delta_1}\mathcal{P}^{l_1}\alpha^{\sigma_1}\beta\cdots\beta\alpha^{\delta_s}\mathcal{P}^{l_s}\alpha^{\sigma_s}$$

and $x = \psi y$. From (2.14), the sufficiency is done . □

Theorem 2.6 A premap is a map if, and only if, its under pregraph is a graph.

Proof From the transitive axiom and Lemma 2.3, its under pregraph is a graph. This is the necessity.

Conversely, from the connectedness and Lemma 2,3 , the premap satisfies the transitive axiom and hence its under pregraph is a graph. This is the sufficiency. □

Example 2.6 In Fig.1.2, the pregraph determined by Par_7 has 2 super premaps:

$$\mathcal{P}_1 = (x_1)(\alpha x_1)(\gamma x_1)(\beta x_1)(x_2, \gamma x_2)(\alpha x_2, \beta x_2)$$

and

$$\mathcal{P}_1 = (x_1)(\alpha x_1)(\gamma x_1)(\beta x_1)(x_2, \beta x_2)(\alpha x_2, \gamma x_2)$$

as shown in Fig.2.1.

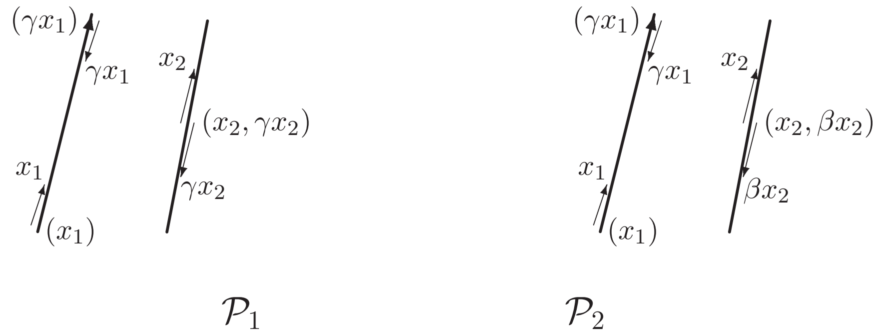
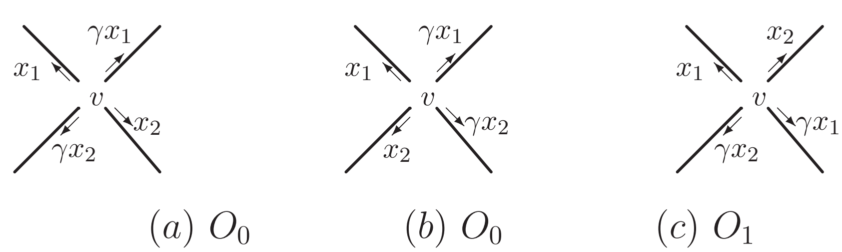


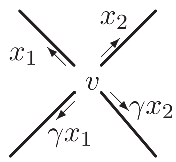
Fig.2.1 Two super premaps

From Theorem 2.6, none of the two super premaps is a map in Fig.2.1.

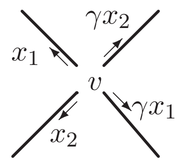
However, the pregraph determined by Par_{12} is a graph in Fig.1.2. From Theorem 2.6, each of its super premaps is a map as shown in the following example.

Example 2.7 In Fig.2.2, there are $2^2 3! = 24$ distinct embeddings of the graph determined by Par_{12} in Fig.1.2. On the associate(or boundary) polygon of the joint tree, the pair of a letter is defined to be of distinct powers when x and γx appear; the same power otherwise.

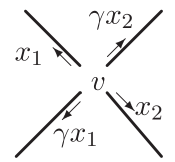




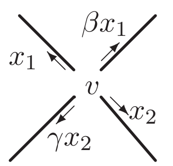
(*d*) O_0



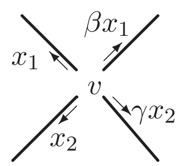
(*e*) O_1



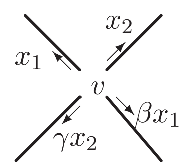
(*f*) O_0



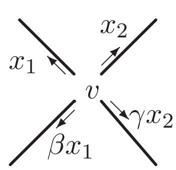
(*g*) N_1



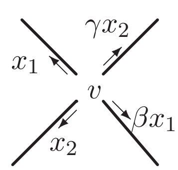
(*h*) N_1



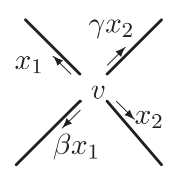
(*i*) N_2



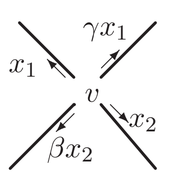
(*j*) N_1



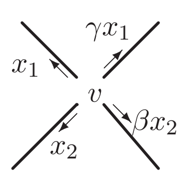
(*k*) N_2



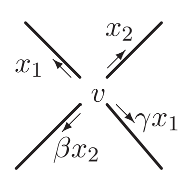
(*l*) N_1



(*m*) N_1



(*n*) N_1



(*o*) N_2

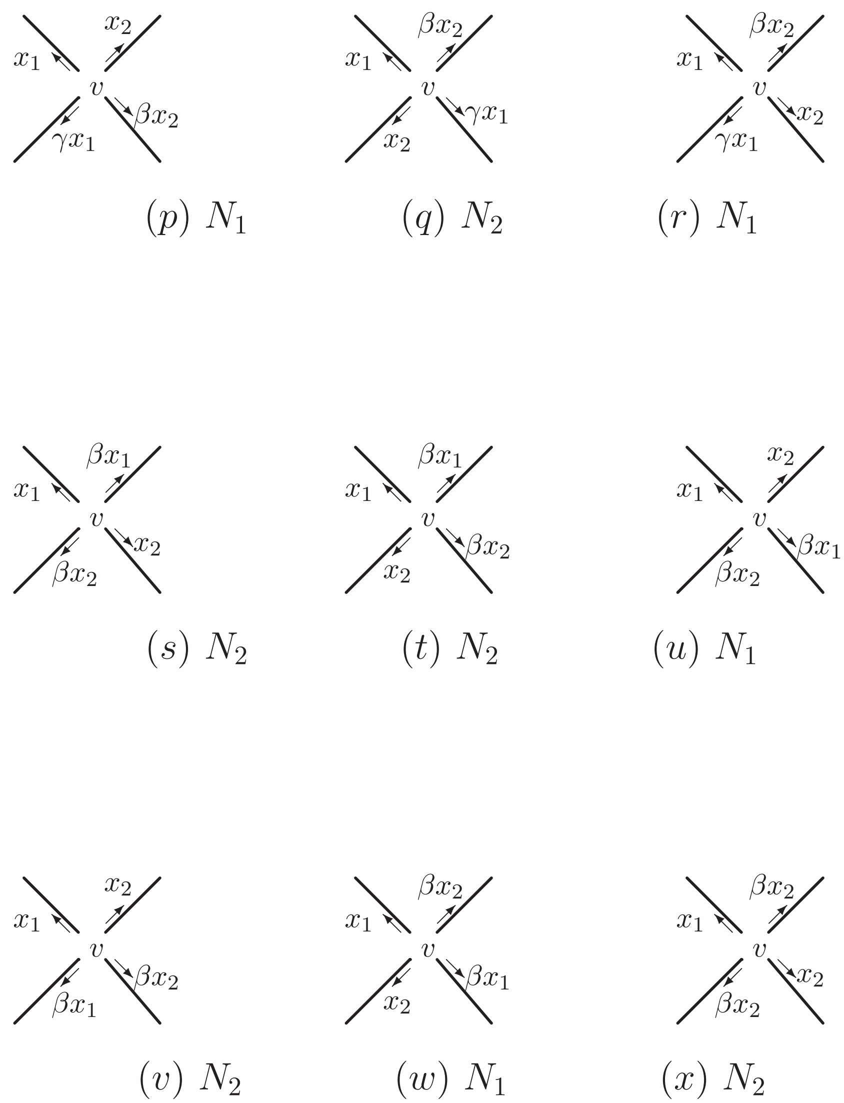


Fig.2.2 All embeddings of a graph

In Fig.2.2, the graph determined by Par_{12}(Fig.1.2) has 6 orientable embeddings $(a$—$f)$. Here, (a), (b), (d) and (f) are the same map on the sphere $O_0 \sim_{\mathrm{top}} (x_1 x_1^{-1})$. And, (c) and (e) are the same map on the torus $O_1 \sim_{\mathrm{top}} (x_1 x_2 x_1^{-1} x_2^{-1})$. Hence, such 6 distinct embeddings are, in fact, 2 maps.

Among the 18 nonorientable embeddings, 10 are on the projective plane and 8 are on the Klein bottle. On the projective plane, (g), (h), (j), (l), (m), (n), (p) and (r) are the same map $(N_1 \sim_{\mathrm{top}} (x_1 x_1^{-1} x_2 x_2))$.

And, (u) and (w) are another map ($N_1 = (x_1x_2x_1x_2)$). On the Klein bottle, (i), (k), (o) and (q) are the same map ($N_2 = (x_1x_2x_1^{-1}x_2)$). And, (s), (t), (v) and (x) are another map ($N_2 \sim_{\text{top}} (x_1x_1x_2x_2)$). Therefore, there are only 4 maps among the 18 embeddings.

II.5 Included angles

Let $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$ be a premap. Write $k = |\mathcal{X}_{\alpha,\beta}(X)/\sim_{\Psi_J}|$, $J = \{\alpha, \beta, \mathcal{P}\}$, and

$$\mathcal{X}_{\alpha,\beta}(X) = \sum_{i=1}^{k} \mathcal{X}_{\alpha_i,\beta_i}(X_i),\ X = \sum_{i=1}^{k} X_i,$$

where $\mathcal{X}_{\alpha_i,\beta_i}(X) \in \mathcal{X}_{\alpha,\beta}/\sim_{\Psi_J}$, α_i and β_i are, respectively, α and β restricted on $\mathcal{X}_{\alpha_i,\beta_i}(X_i)$, $i = 1, 2, \cdots, k$. Further,

$$M = \sum_{i=1}^{k} M_i,\ M_i = (\mathcal{X}_{\alpha_i,\beta_i}(X_i), \mathcal{P}_i), \tag{2.16}$$

where M_i is a map and $\mathcal{P}_i$ is $\mathcal{P}$ restricted on $\mathcal{X}_{\alpha_i,\beta_i}(X_i)$, $i = 1, 2, \cdots, k$.

This enables us only to discuss maps without loss of generality.

Lemma 2.4 Any map $(\mathcal{X}, \mathcal{P})$ has that $(\mathcal{P}\alpha)^2 = 1$.

Proof From the conjugate axiom,

$$\begin{aligned}(\mathcal{P}\alpha)^2 &= (\mathcal{P}\alpha)(\mathcal{P}\alpha) = \mathcal{P}(\alpha\mathcal{P})\alpha \\ &= (\mathcal{P}\mathcal{P}^{-1})(\alpha\alpha) = 1.\end{aligned}$$

This is the conclusion of the lemma. □

Lemma 2.5 Any map $(\mathcal{X}, \mathcal{P})$ has that $(\mathcal{P}^*\beta)^2 = 1$.

Proof Because $\mathcal{P}^*\beta = \mathcal{P}\gamma\beta = \mathcal{P}\alpha(\beta\beta) = \mathcal{P}\alpha$, from Lemma 2.4, the conclusion is obtained. □

On the basis of the above two lemmas, on a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, any $x \in \mathcal{X}_{\alpha,\beta}$ has that

$$(x, \mathcal{P}\alpha x) = (\mathcal{P}^*\beta x, x). \tag{2.17}$$

Thus, $\langle x, \mathcal{P}\alpha x\rangle$, or $\langle \mathcal{P}^*\beta x, x\rangle$, is called an *included angle* of the map. For an edge $Kx = \{x, \alpha x, \beta x, \gamma x\}$ of M, $\{x, \alpha x\}$ and $\{\beta x, \gamma x\}$ are its two *ends* , or *semiedges*. And, $\{x, \beta x\}$ and $\{\alpha x, \gamma x\}$ are its two *sides*, or *cosemiedges*.

Theorem 2.7 For a $(\mathcal{X}(X), \mathcal{P})$, let $V = \{v|v = \{x\}_{\mathcal{P}} \cup \{\alpha x\}_{\mathcal{P}}, \forall x \in X\}$ and $F = \{f|f = \{x\}_{\mathcal{P}^*} \cup \{\beta x\}_{\mathcal{P}^*}, \ \forall x \in X\}$. If x_v and x_f are, respectively, in v and f as representatives, then

$$\begin{aligned}\mathcal{X} &= \sum_{v\in V}(\{x_v, \alpha\mathcal{P}^{-1}x_v\} + \{\mathcal{P}x_v, \alpha x_v\} + \cdots \\ &\qquad + \{\mathcal{P}^{-1}x_v, \alpha\mathcal{P}^{-2}x_v\}) \\ &= \sum_{f\in F}(\{x_f, \beta\mathcal{P}^{*-1}x_f\} + \{\mathcal{P}^*x_f, \beta x_f\} + \cdots \\ &\qquad + \{\mathcal{P}^{*-1}x_f, \beta\mathcal{P}^{*-2}x_f\}).\end{aligned} \tag{2.18}$$

Proof From $\mathcal{X} = \sum_{v\in V} v$ and the conjugate axiom,

$$\begin{aligned}v &= \{x_v, \mathcal{P}\alpha x_v\} + \{\mathcal{P}x_v, \mathcal{P}\alpha\mathcal{P}x_v\} + \cdots \\ &\qquad + \{\mathcal{P}^{-1}x_v, \mathcal{P}\alpha\mathcal{P}^{-1}x_v\} \\ &= \{x_v, \alpha\mathcal{P}^{-1}x_v\} + \{\mathcal{P}x_v, \alpha x_v\} + \cdots \\ &\qquad + \{\mathcal{P}^{-1}x_v, \alpha\mathcal{P}^{-2}x_v\}.\end{aligned}$$

This is the first equality.

The second equality can similarly be derived from $\mathcal{X} = \sum_{f\in F} f$ and Lemma 2.1(the conjugate axiom for $\mathcal{P}^*$ with β). □

It is seen from the theorem that the numbers of included angles, semiedges and co-semiedges are, each, equal to the sum of degrees of vertices. Since every edge has exactly 2 semiedges, this number is 2 times the size of the map.

Activities on Chapter II

II.6 Observations

O2.1 For a set of $4k$ elements, observe how many ground sets can be produced.

O2.2 By a set of 12 elements as an example, observe how many basic permutations satisfy the conjugate axiom.

O2.3 In Example 2, observe which nonbasic partition has a permutation with the conjugate axiom.

O2.4 Provide two embeddings with the same under graph.

O2.5 For an embedding, observe if its mirror image is the same as itself. How about for a map?

O2.6 For a map $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, observe the orbits of permutations $\mathcal{P}\alpha$ and $\alpha\mathcal{P}$. Whether, or not, they are a map.

O2.7 For a map $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, observe the orbits of permutations $\mathcal{P}\beta$ and $\beta\mathcal{P}$. Whether, or not, they are a map.

O2.8 On a map $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, whether, or not, the permutation $\mathcal{P}\alpha\beta$ is a map on the same ground set.

A map with each of its face a quadrangle is called a *quadrangulation*. A map with only triangular faces is a *triangulation*.

In general, a map with all vertices of the same degree is said to be *vertex regular*, or *primal regular*. If all faces are of the same degree, the the map is said to be *face regular* , or *dual regular*. A primal regular map with its vertex degree i, $i \geq 1$, is called an *i-map*. A

dual regular map with its face degree j, $j \geq 1$, is called a j^*-*map*. A triangulation and a quadrangulation are, respectively, a 3^*-map and a 4^*-map in their own right.

O2.9 Whether, or not, the under graph of a 4^*-map is always bipartite. Furthermore, whether, or not, the under graph of a $(2k)^*$-map, $k \geq 3$, is always bipartite.

O2.10 Observe that for any integer i, $i \geq 1$, whether, or not, there always exists an i-map and an i^*-map.

If the degree of any vertex(or face) of a map is only an integer i, or j, $j \neq i$, $i, j \geq 1$, then the map is called a (i, j)-*map* (or (i^*, j^*)-*map*). Similarly, the meanings of a (i, j^*)-map and a (i^*, j)-map are known.

O2.11 For any i, j, $i \neq j \geq 1$, whether, or not, there is a (i, j)-map, a (i^*, j^*)-map, or a (i, j^*)-map.

II.7 Exercises

E2.1 For two permutations Per_1 and Per_2 on a set of 4 elements with ${\mathrm{Per}_1}^2 = {\mathrm{Per}_2}^2 = 1$, list all the generated groups $\Psi_{\{\mathrm{Per}_1, \mathrm{Per}_2\}}$. Show if each of them is isomorphic to the Klein group.

E2.2 Let λ and μ be permutations on the set $\mathcal{A}$ and $\lambda^2 = \mu^2 = 1$. Prove that if the generated group $\Psi_{\{\lambda,\mu\}}$ is isomorphic to the Klein group, then $\mathcal{A}$ is a ground set if, and only if, $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \cdots + \mathcal{A}_k$, $k \geq 1$, such that $\lambda|_{\mathcal{A}_i}$ and $\mu|_{\mathcal{A}_i}$ are both have two orbits, and $\lambda|_{\mathcal{A}_i} \neq \mu|_{\mathcal{A}_i}$, $1 \leq i \leq k$.

E2.3 For a map $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, prove that $\alpha(x)_{\gamma\mathcal{P}} = (\alpha x)^{-1}_{\mathcal{P}\gamma}$.

E2.4 For a map $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, prove that $(x)_{\gamma\mathcal{P}}$ and $(\beta x)_{\gamma\mathcal{P}}$ are conjugate.

E2.5 To prove that all planar embeddings of K_4, *i.e.*, the complete graph of order 4, are the same map.

A map is said to be *nonseparable* if its under graph is nonsepa-

rable, *i.e.* , no *cut-vertex*(its deletion with incident edges destroys the connectedness in a graph).

E2.6 List all nonseparable maps of size 4.

A map with all vertices of even degree is called a *Euler map.* If the face set of a map can be partitioned into two parts each of which no two faces have an edge in common, then the face partition is called a *edge independent* 2-partition .

E2.7 Provide and prove a condition for the face set of a Euler map having an edge independent 2-partition.

E2.8 Prove the following statements:

(i) The permutation β on the ground set $\mathcal{X}$ is a map if, and only if, $\mathcal{X} = Kx$;

(ii) The permutation α on the ground set $\mathcal{X}$ is a map if, and only if,

$$\mathcal{X} = K^*x = \{x, \beta x, \alpha x, \gamma x\},$$

i.e., β is the first operation.

E2.9 For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$,

(i) Provide a map M and an integer i, $i \geq 2$, such that $\mathcal{P}^i$ is not a map;

(ii) For i, $i \geq 2$, provide the condition such that $\mathcal{P}^i$ is still a map.

E2.10 Let $\mathcal{C}$ and $\mathcal{D}$ be, respectively, the sets of 3-maps and 3^*-maps. For the size given, provide a 1–to–1 correspondence between them.

II.8 Researches

From Theorem 1.10 in Chapter I, any graph has an embedding on a surface (orientable or nonorientable). However, if an embedding is restricted to a particular property, then the existence is still necessary to investigate. If a map has each of its faces partitionable into circuits,

then it is called a *favorable map*. If a graph has an embedding which is a favorable map, then the embedding is also said to be *favorable*[Liu12].

R2.1 Conjecture. Any graph without cut-edge has a favorable embedding.

It is easily checked and proved that a graph with a cut-edge does not have a favorable embedding. However, no graph without cut-edge is exploded to have no favorable embedding yet. Some types of graphs have be shown to satisfy this conjecture such as K_n, $n \geq 3$; $K_{m,n}$, $m, n \geq 2$, Q_n, $n \geq 2$, planar graphs without cut-edge *etc*.

A map which has no face itself with a common edge is said to be *preproper* . It can be shown that all preproper maps are favorable. However, the converse case is unnecessary to be true.

R2.2 Conjecture. Any graph without cut-edge has a preproper embedding.

Similarly, it is also known that any graph with a cut-edge does not have a preproper embedding. And, K_n, $n \geq 3$; $K_{m,n}$, $m, n \geq 2$, Q_n, $n \geq 2$, planar graphs without cut-edge *etc* are shown to satisfy the conjecture as well.

Furthermore, if a map has each of its faces a circuit itself, then it is called a *proper map*, or *strong map*. Likewise, *proper embedding*, or *strong embedding* . It can be shown that all proper maps are preproper. However, the converse case is unnecessary to be true.

R2.3 Conjecture. Any graph without cut-edge has a proper embedding.

For proper embeddings as well, it is known that any graph with a cut-edge does not have a proper embedding. And, K_n, $n \geq 3$; $K_{m,n}$, $m, n \geq 2$, Q_n, $n \geq 2$, planar graphs without cut-edge *etc* are all shown to satisfy this conjecture.

Although conjectures R2.1—R2.3 are stronger to stronger, because R2.3 has not yet shown to be true, or not, the the two formers are still meaningful.

If a favorable(proper) embedding of a graph of order n has at

most $n-1$ faces, then it said to be of *small face* .

Now, it is known that triangulations on the sphere have a small face proper embedding. Because triangulations of order n have exactly $3n-6$ edges and $2n-4$ faces, all the small face embeddings are not yet on the sphere for $n \geq 4$.

R2.4 Conjecture. Any graph of order at least six without cut-edge has a small face proper embedding.

Because it is proved that K_5 has a proper embedding only on the surfaces of orientable genus 1(torus) and nonorientable genus at most 2(Klein bottle)[WeL1], they have at least 5 faces and hence are not of small face.

R2.5 Conjecture. Any nonseparable graph of order n has a proper embedding with at most n faces.

If a map only has i-faces and $i+1$-faces, $3 \leq i \leq n-1$, then it is said to be *semi-regular.*

R2.6 Conjecture. Any nonseparable graph of order n, $n \geq 7$, has a semi-regular proper embedding.

In fact, if a graph without cut-edge has a cut-vertex, then it can be decomposed into nonseparable blocks none of which is a link itself. If this conjecture is proved, then it is also right for a graph without cut-edge. Some relationships among these conjectures and more with new developments can be seen in [Liu12].

R2.7 For an integer $i \geq 3$, provide a necessary and sufficient condition for a graph having an i-embedding, or i^*-embedding. Particularly, when $i = 3, 4$ and 5.

First, start from $i = 3$ with a given type of graphs. For instance, choose $G = K_n$, the complete graph of order n. For 3^*-embedding, on the basis of Theorem 1.12(called *Euler formula*), a necessary condition for K_n, $n \geq 3$, having an 3^*-embedding is

$$n - \frac{n(n-1)}{2} + \phi = 2 - 2p \text{ and } 3\phi = n(n-1),$$

where ϕ and p are, respectively, the face number of an 3^*-embedding

and the genus of the orientable surface the embedding is on. It is known that the condition is still sufficient.

If nonorientable surfaces are considered, the necessary condition

$$n - \frac{n(n-1)}{2} + \phi = 2 - q \text{ and } 3\phi = n(n-1),$$

i.e.,

$$q = \frac{(n-3)(n-4)}{6},$$

where q is the nonorientable genus of the surface an 3^*-embedding is on is not sufficient anymore for $q \geq 1$. because when $n = 7$, K_7 would have an 3^*-embedding on the surface of nonorientable genus $q = 2$. However, it is shown that K_7 is not embeddable on the Klein bottle([Lemma 4.1 in [Liu11]). It has been proved that except only for this case, the necessary condition is also sufficient.

More other types of graphs, such as the complete bipartite graph $K_{m,n}$, n-cube Q_n and so forth can also be seen in [Liu11].

R2.8 For $3 \leq i, j \leq 6$, recognize if a graph has an (i, j)-embedding (or (i^*, j^*)-embedding).

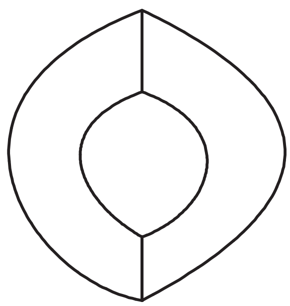
More generally, investigate the upper or/and the lower bounds of i and j such that for a given type of graphs having an (i^*, j^*)-embedding.

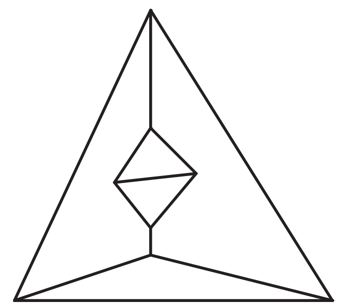
R2.9 Given two integers i, j not less than 7, justify if a graph has an (i, j)-embedding(or (i^*, j^*)-embedding).

If a proper map has any pair of its faces with at most 1 edge in common, then it is called a *polygonal map* .

R2.10 Conjecture. Any 3-connected graph has an embedding which is a polygonal map.

From (a) and (b) in Fig.2.3, this conjecture is not valid for non-separable graphs. The two graphs are nonseparable. The graph in (a) has a multi-edge, but that in (b) does not. It can be checked that, none of them has a polygonal embedding.





(a) (b)

Fig.2.3 Two graphs without polygonal embedding

If an embedding of a graph has its genus(orientable or nonorientable) minimum among all the embeddings of the graph, then it is called a *minimum* (*orientable or nonorientable*) *genus embedding*. Based on the Euler formula, a minimum genus embedding has its face number maximum. So, a minimum (orientable or nonorientable)genus embedding is also called a *maximum* (*orientable or nonorientable*) *face number embedding*. Maximum face number implies that the average length of faces is smaller, and hence the possibility of faces being circuits is greater. This once caused to guess that minimum genus embeddings were all proper. However, a nonproper minimum genus embedding of a specific graph can be constructed by making 1 face as greater as possible with all other faces as less as possible. In fact, for torus and projective plane, all maximum face number embeddings are shown to be proper. For surfaces of big genus, a specific type of graphs were provided for all of their maximum face number embeddings nonproper[Zha1].

R2.11 Conjecture. Any nonseparable regular graph has a maximum face number embedding which is a proper map.

A further suggestion is to find an embedding the lengths of all faces are nearly equal. The difference between the maximum length

and the minimum length of faces in a map is call the *equilibrium* of the map. An embedding of a graph with its equilibrium minimum is called an *equilibrious embedding* of the graph.

R2.12 Conjecture. Any 3-connected graph has an equilibrious embedding which is proper.

An approach to access the conjecture is still for some types of graphs, *e.g.*, planar graphs, Halin graphs, Hamiltonian graphs, further graphs embeddable to a surface with given genus *etc.*

Chapter III

Duality

- The dual of a map $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is the map $(\mathcal{X}_{\beta,\alpha}, \mathcal{P}\alpha\beta)$ and vice versa.
- The deletion of an edge in a map is the contraction of the corresponding edge in the dual map and vice versa.
- The addition of an edge to, the inverse of deleting an edge in, a map is splitting off its corresponding edge on, the inverse of contracting an edge in, the dual map and vice versa.
- The deletion of an edge with its inverse, the addition, and the dual of deletion, the contraction of an edge with its inverse, splitting off an edge are restricted on the same surface to form basic transformations.

III.1 Dual maps

On the basis of II.2, for a basic permutation $\mathcal{P}$ on the ground set $\mathcal{X}_{\alpha,\beta}$, $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is a premap if, and only if, $(\mathcal{X}_{\beta,\alpha}, \mathcal{P}^*)$ is a premap where $\mathcal{P}^* = \mathcal{P}\gamma$, $\gamma = \alpha\beta$(Theorem 2.4). The latter is called the *dual* of the former. Since

$$\mathcal{P}^{**} = \mathcal{P}^*\beta\alpha = (\mathcal{P}\alpha\beta)\beta\alpha = \mathcal{P}(\alpha\beta\beta\alpha) = \mathcal{P},$$

the former is also the dual of the latter.

Because the transitivity of two elements in the ground set $\mathcal{X}_{\alpha,\beta}$ on a premap $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ under the group Ψ_J, $J = \{\mathcal{P}, \alpha, \beta\}$, determine an equivalence, denoted by $\sim_{\Psi_J}$, the restriction of $\mathcal{P}$ on a class

$$\mathcal{X}_{\alpha,\beta}/\sim_{\Psi_J}$$

is called a *transitive block*.

Theorem 3.1 Premap $M_2 = (\mathcal{X}, \mathrm{Per}_2)$ is the dual of premap $M_1 = (\mathcal{X}, \mathrm{Per}_1)$ if, and only if,

$$\mathcal{X}/\sim_{\Psi_{J_2}} = \mathcal{X}/\sim_{\Psi_{J_1}}, \tag{3.1}$$

where

$$J_2 = \{\mathrm{Per}_2, \alpha, \beta\},\ J_1 = \{\mathrm{Per}_1, \alpha, \beta\},$$

and

$$\mathrm{Per}_2 = \mathrm{Per}_1\gamma = \mathrm{Per}_1{}^*,\ \gamma = \alpha\beta.$$

Proof Necessity. Since M_2 is the dual of M_1, $\mathrm{Per}_2 = \mathrm{Per}_1\alpha\beta$. From $\mathrm{Per}_2 = \mathrm{Per}_1\alpha\beta \in \Psi_{J_1}$, $\Psi_{J_2} = \Psi_{J_1}$. Hence, (3.1) holds. This is the necessity.

Sufficiency. Since M_1 is a premap and $\mathrm{Per}_2 = \mathrm{Per}_1\alpha\beta$, M_2 is also a premap, and then the dual of M_1 by considering Theorem 2.4. This is the sufficiency. $\square$

From this theorem, the duality between M_1 and M_2 induces a 1–to–1 correspondence between their transitive blocks in dual pair. Because each transitive block is a map, it leads what the *dual map* of a map is. The representation of a premap by its transitive blocks is called its *transitive decomposition*.

Example 3.1 Map

$$\tilde{L}_1 = (\{x, \alpha x, \beta x, \gamma x\}, (x, \beta x)),\ \gamma = \alpha\beta,$$

and its dual

$$\tilde{L}_1^* = (\{x, \beta x, \alpha x, \gamma x\}, (x, \alpha x)).$$

Or, in the form as

$$\tilde{L}_1 = (e, v) \text{ and } \tilde{L}_1^* = (e^*, f),$$

where $v = (x, \beta x),\ f = (x, \alpha x)$,

$$e = \{x, \alpha x, \beta x, \gamma x\},\ \gamma = \alpha\beta,$$

and

$$e^* = \{x, \beta x, \alpha x, \gamma x\},$$

as shown in Fig.3.1.

In the following figure, two As and two Bs are, respectively, identified on the surface.

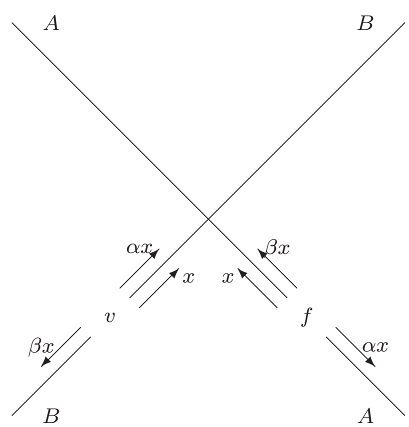


Fig.3.1 Map and its dual

This figure shows what a dual pair of maps looks like. It is a generalization of a dual pair of maps on the plane.

A map with its under pregraph a selfloop is called a *loop map*. It is seen that $\tilde{L}_1$ and its dual $\tilde{L}_1^*$ in Fig.3.1 are both loop maps.

In a premap $M = (\mathcal{X}, \mathcal{P})$, if a vertex $v = \{(x)_{\mathcal{P}},\ (\alpha x)_{\mathcal{P}}\}$ is transformed into two vertices

$$\begin{aligned} v_1 &= \{(x,\ \mathcal{P}x,\ \cdots, \mathcal{P}^j x),\ (\alpha x,\ \alpha\mathcal{P}^j x,\ \cdots, \alpha\mathcal{P}x)\} \\ &= \{(x)_{\mathcal{P}'}, (\alpha x)_{\mathcal{P}'}\} \end{aligned}$$

and

$$v_2 = \{(\mathcal{P}^{j+1}x, \mathcal{P}^{j+2}x, \cdots, x\mathcal{P}^{-1}x), (\alpha\mathcal{P}^{j+1}x, \alpha\mathcal{P}^{-1}x, \cdots, \alpha\mathcal{P}^{j+2}x)\}$$
$$= \{(\mathcal{P}^{j+1}x)_{\mathcal{P}'}, (\alpha\mathcal{P}^{j+1}x)_{\mathcal{P}'}\},\ j \geq 0,$$

with other vertices unchanged for permutation $\mathcal{P}$ becoming permutation $\mathcal{P}'$. It is seen that permutation $\mathcal{P}'$ is basic and with the conjugate axiom as well. Hence, $M' = (\mathcal{X}, \mathcal{P}')$ is also a premap. Such an operation is called *cutting* a vertex. If elements at v_1 are not transitive with elements at v_2 in M', then elements at v_1 and elements at v_2 are said to be *cuttable* in M. The vertex v is called a *cutting vertex* in M; otherwise, *noncuttable* .

If there are two elements cuttable in a map, the the map is said to be cuttable in its own right.

In virtue of Theorem 3.1, cuttability and noncuttability are concerned with only maps without loss of the generality of premaps.

Lemma 3.1 A map M is cuttable if, and only if, its dual M^* is cuttable.

Proof Necessity. From $M = (\mathcal{X}, \mathcal{P})$ cuttable, vertex

$$v = (x,\ \mathcal{P}x,\ \cdots, \mathcal{P}^{-1}x)$$

is assumed to cut into two vertices as

$$v_1 = (x,\ \mathcal{P}x,\ \cdots, \mathcal{P}^{j}x)$$

and

$$v_2 = (\mathcal{P}^{j+1}x, \mathcal{P}^{j+2}x, \cdots, \mathcal{P}^{-1}x)$$

for obtaining premap $M' = (\mathcal{X}, \mathcal{P}') = M_1 + M_2$ where v_i is on $M_i = (\mathcal{X}_i, \mathcal{P}_i)$, $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$, $\mathcal{P}_i$ is the restriction of $\mathcal{P}'$ on $\mathcal{X}_i$,$i = 1, 2$. It can be checked that M_1 and M_2 are both maps. Thus, on $M^* = (\mathcal{X}, \mathcal{P}^*)$, vertex

$$v^* = (x, A, \gamma\mathcal{P}^{j}x, \mathcal{P}^{j+1}x, B, \gamma\mathcal{P}^{-1}x)$$

can be cut into two vertices

$$v_1^* = (x, A, \gamma\mathcal{P}^{j}x)$$

and

$$v_2^* = (\mathcal{P}^{j+1}x, B, \gamma\mathcal{P}^{-1}x)$$

where A and B are, respectively, linear orders of elements in $\mathcal{X}_1$ and $\mathcal{X}_2$. This attains the premap $M^{*\prime} = (\mathcal{X}, \mathcal{P}^{*\prime}) = M_1^* + M_2^*$ where $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$, ${\mathcal{P}^*}_i$ is the restriction of $\mathcal{P}^{*\prime}$ on $\mathcal{X}_i$,$i = 1, 2$. The necessity is obtained.

Sufficiency. From the duality, it is deduced from the necessity.□

If two elements in the ground set of a map M are not transitive in M' obtained by cutting a vertex on M, they are said to be *cuttable*; otherwise, *noncuttable*. It can be checked that the noncuttability determines an equivalence, denoted by $\sim_{\rm nc}$ on the ground set of M. The restriction of M on each

$$\mathcal{X}_{\alpha,\beta}/\sim_{\rm nc}$$

is called a *noncuttable block*.

If all noncuttable blocks and all cutting vertices of a map deal with vertices such that two vertices are adjacent if, and only if, one is a noncuttable block and the other is a cutting vertex incident to the block, then the graph obtained in this way is called a *cutting graph* of the map. It is easily shown that the cutting graph of a map is always a tree.

For a face $f = (x)_{\mathcal{P}\gamma}$ of a premap $M = (\mathcal{X}, \mathcal{P})$, if there has, and only has, an integer $l \geq 0$ for transforming f into

$$f_1 = (x, \cdots, (\mathcal{P}\gamma)^l x)$$

and

$$f_2 = ((\mathcal{P}\gamma)^{l+1}x, \cdots, (\mathcal{P}\gamma)^{-l}x)$$

such that x and $(\mathcal{P}\gamma)^{l+1}x$ are not transitive at all, then f is called a *cutting face* of M. From the procedure in the proof of Lemma 3.1, For a cutting vertex of a premap, there has, and only has, a corresponding cutting face in the dual of the premap.

Theorem 3.2 Two maps M and N are mutually dual if, and only if, their cutting graph are the same and the corresponding non-

cuttable blocks are mutually dual such that a cutting vertex of one corresponds to a cutting face of the other.

Proof Necessity. Because maps M and N are mutually dual, from the procedure in the proof of Lemma 3.1, there is a 1–to–1 correspondence between their noncuttable blocks such that two corresponding blocks are mutually dual. There is also a 1–to–1 correspondence between their cutting vertices such that the cyclic orders of their blocks at two corresponding cutting vertices are in correspondence. Therefore, their cutting graphs are the same. This is the necessity.

Sufficiency. Because maps M and N have the same cutting graph, a tree of course, in virtue of the correspondence between cutting vertices and cutting faces, the sufficiency is deduced from Lemma 3.1. $\square$

Note 3.1 Tow trees are said to be the same in the theorem when trees as maps(planar of course) are the same but not the isomorphism of trees as graphs(the latter can be deduced from the former but unnecessary to be true from the latter to the former).

On a premap $M = (\mathcal{X}, \mathcal{P})$, if an edge Kx is incident with two faces, *i.e.*,

$$\gamma x \notin \{x\}_{\mathcal{P}\gamma} \cup \{\beta x\}_{\mathcal{P}\gamma},$$

then it is said to be *single* ; otherwise, *i.e.*,

$$\gamma x \in \{x\}_{\mathcal{P}\gamma} \cup \{\beta x\}_{\mathcal{P}\gamma}$$

(with only one face), *double*. An edge with distinct ends is called a *link*; otherwise, a *loop* . Clearly, *single link*, *single loop*, *double link* and *double loop*. Further, a double link is called a *harmonic link*, or *singular link* according as $\gamma x \in (x)_{\mathcal{P}\gamma}$, or not. Similarly, a single loop is called a *harmonic loop*, or *singular loop* according as $\gamma x \in (x)_{\mathcal{P}\gamma}$, or not.

Theorem 3.3 For an edge $e_x = \{x,\ \alpha x, \beta x, \gamma x\}$ of premap $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ and its corresponding edge $e^*_x = \{x,\ \beta x, \alpha x, \gamma x\}$ of the dual $M^* = (\mathcal{X}_{\beta,\alpha}, \mathcal{P}^*)$, $\mathcal{P}^* = \mathcal{P}\gamma$,

(i) e_x is a single link if, and only if, e^* is a single link;

(ii) e_x is a harmonic link if, and only if, e^* is a harmonic loop;

(iii) e_x is a singular link if, and only if, e^* is a singular loop;

(iv) e_x is a double loop if, and only if, e^* is a double loop.

Proof Necessity. (i) Because e_x is a link, $(x)_{\mathcal{P}}$ and $(\gamma x)_{\mathcal{P}}$ belong to distinct vertices. And because e_x is a single edge, $(x)_{\mathcal{P}_\gamma}$ and $(\gamma x)_{\mathcal{P}_\gamma}$ belong to distinct face. By the duality, e^*_x is a single link as well.

(ii) Because e_x is a double link, in spite of $(x)_{\mathcal{P}}$ and $(\gamma x)_{\mathcal{P}}$ belonging to distinct vertices, γx, or $\alpha x \in (x)_{\mathcal{P}^*}$. And because of harmonic link, the only opportunity is $\gamma x \in (x)_{\mathcal{P}^*}$. From the duality, e^*_x is a harmonic loop.

(iii) Because e_x is a singular link, in spite of $(x)_{\mathcal{P}}$ and $(\gamma x)_{\mathcal{P}}$ belonging to distinct vertices, $\alpha x \in (x)_{\mathcal{P}^*}$. In virtue of α as the second operation of M^*, e^*_x is a singular loop.

(iv) Because e_x is a double loop, $\beta x \in (x)_{\mathcal{P}}$ and $\alpha x \in (x)_{\mathcal{P}^*}$. From the symmetry between α and β, M and M^*, e^*_x is a double loop as well.

Sufficiency. From the symmetry in duality, *i.e.*, $M = (M^*)^*$, it is obtained from the necessity. $\square$

On the basis of this theorem, the classification and the dual relationship among edges are shown in Table 3.1.

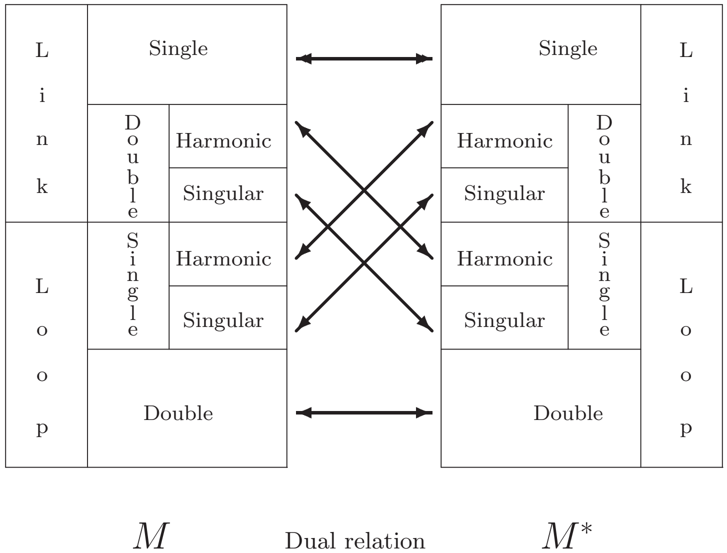


Table 3.1 Duality between edges

In the table above, harmonic links will be classified into segmentation edges and terminal links and harmonic loops into shearing loops and terminal loops in III.2 to have additional two dual pairs of edges: segmentation edges and shearing loops, terminal links and terminal loops.

III.2 Deletion of an edge

Let $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$be a premap and $e_x = Kx = \{x, \alpha x, \beta x, \gamma x\}$, $x \in X$, an edge.

What is obtained by *deleting* the edge e_x from M is denoted by

$$M - e_x = (\mathcal{X}_{\alpha,\beta}(X) - Kx, \mathcal{P}_{-x}) \tag{3.2}$$

where $\mathcal{P}_{-x}$ is the permutation restricted from $\mathcal{P}$ on $\mathcal{X}_{\alpha,\beta}(X) - Kx$.

Lemma 3.2 Permutation $\mathcal{P}_{-x}$ is determined in the following

way as when e_x is not a selfloop,

$$\mathcal{P}_{-x}y = \begin{cases} \mathcal{P}x(\text{and } \alpha\mathcal{P}^{-1}x), \\ \quad \text{if } y = \mathcal{P}^{-1}x(\text{and } \alpha\mathcal{P}x); \\ \mathcal{P}\gamma x(\text{和 } \alpha\mathcal{P}^{-1}\gamma x), \\ \quad \text{if } y = \mathcal{P}^{-1}\gamma x(\text{and } \alpha\mathcal{P}\gamma x); \\ \mathcal{P}y, \text{ otherwise,} \end{cases} \tag{3.3}$$

and when e_x is a selfloop with $\gamma x \in (x)_{\mathcal{P}}$,

$$\mathcal{P}_{-x}y = \begin{cases} \mathcal{P}x(\text{and } \alpha\mathcal{P}^{-1}x), \\ \quad \text{if } y = \mathcal{P}^{-1}x(\text{and } \alpha\mathcal{P}x); \\ \mathcal{P}\gamma x(\text{and } \alpha\mathcal{P}^{-1}\gamma x), \\ \quad \text{if } y = \mathcal{P}^{-1}x(\text{and } \alpha\mathcal{P}\gamma x); \\ \mathcal{P}y, \text{ otherwise,} \end{cases} \tag{3.4}$$

otherwise, *i.e.*, $\gamma x \notin (x)_{\mathcal{P}}$, γx is replaced by βx in (3.4).

Proof When e_x is not a selfloop. Because only vertices $(x)_{\mathcal{P}}$ and $(\gamma x)_{\mathcal{P}}$ are, respectively, changed in $M - e_x$ from M as

$$(\mathcal{P}^{-1}x)_{\mathcal{P}_{-x}} = (\mathcal{P}^{-1}x, \mathcal{P}^2x, \cdots, \mathcal{P}^{-2}x)$$

and

$$(\mathcal{P}^{-1}\gamma x)_{\mathcal{P}_{-x}} = (\mathcal{P}^{-1}\gamma x, \mathcal{P}\gamma x, \cdots, \mathcal{P}^{-2}\gamma x)$$

(Fig.3.2(a)⇒(b)). This implies (3.3).

When e_x is a selfloop with $\gamma x \in (x)_{\mathcal{P}}$. Because only vertex $(x)_{\mathcal{P}}$ is changed in $M - e_x$ from M as

$$(\mathcal{P}^{-1}x)_{\mathcal{P}_{-x}} = (\mathcal{P}^{-1}x, \mathcal{P}x, \cdots, \mathcal{P}^{-1}\gamma x, \mathcal{P}\gamma x, \cdots, \mathcal{P}^{-2}x)$$

(Fig.3.2(c)→(d)), or

$$(\mathcal{P}^{-1}x)_{\mathcal{P}_{-x}} = (\mathcal{P}^{-1}x, \mathcal{P}x, \cdots, \mathcal{P}^{-1}\beta x, \mathcal{P}\beta x, \cdots, \mathcal{P}^{-2}x)$$

(Fig.3.2(c)→(d) in parentheses) according as $\gamma x \in (x)_{\mathcal{P}}$, or not.

The former is (3.4). The latter is what is obtained from (3.4) with γx is replaced by βx. □

In Fig.3.2, the left two figures are parts of the original map and the right two figures, the results by deleting the edge Kx.

Further, Fig.3.3–9 are all like this without specification.

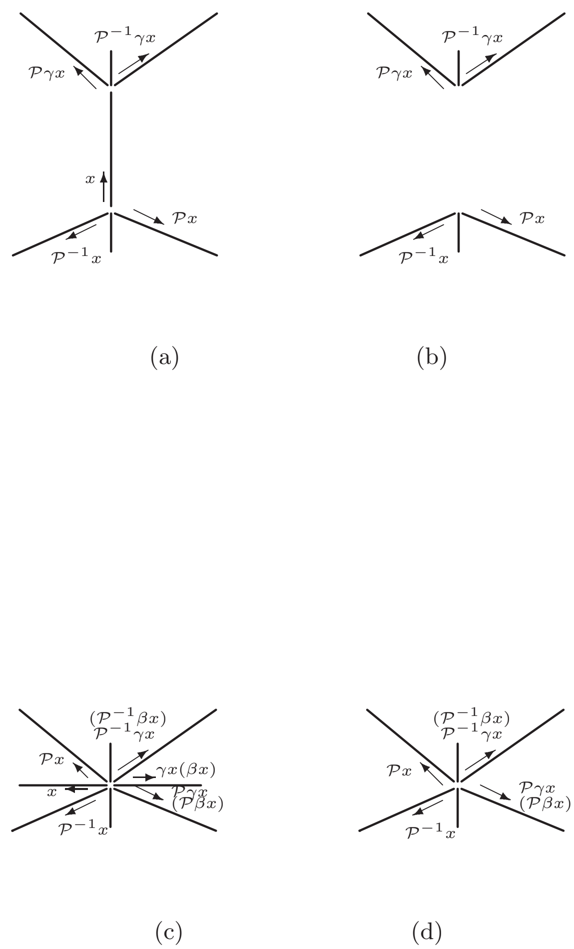


Fig.3.2 Deletion of an edge

Lemma 3.3 For a premap $M = (\mathcal{X}, \mathcal{P})$, $M - e_x = (\mathcal{X} -$

$Kx, \mathcal{P}_{-x})$ is also a premap. And, the number of transitive blocks in $M - e_x$ is not less than that in M.

Proof Because $\mathcal{P}$ is basic for α, from Lemma 3.2 $\mathcal{P}_{-x}$ is also basic for α. Because $\mathcal{P}$ satisfies the conjugate axiom for α, from Lemma 3.2 and Theorem 2.3 $\mathcal{P}_{-x}$ is also satisfies the conjugate axiom for α. The first statement is done.

Because any nontransitive pair of elements in M is never transitive in $M - e_x$, the second statement is done. $\square$

If e_x is an edge of a premap M such that $M - e_x$ has more transitive blocks than M does, then e_x is called a *segmentation edge*. If an edge has its one end formed by only one semiedge of the edge itself, then it is called an *terminal link*. From the symmetry of elements in a quadricell, (x) can be assumed as the 1-vertex incident with a terminal link without loss of generality. Since $(\mathcal{P}\gamma)\gamma x = \mathcal{P}x = x$, $\gamma x \in (x)_{\mathcal{P}\gamma}$. Hence, a terminal link is always a harmonic link. However, a harmonic link is unnecessary to be a terminal link. This point can be seen in the following theorem.

Theorem 3.4 For a map $M = (\mathcal{X}, \mathcal{P})$, $M - e_x = (\mathcal{X} - Kx, \mathcal{P}_{-x})$ is a map if, and only if, e_x is not a harmonic link of M except for terminal link.

Proof When $\langle x, \gamma x\rangle \subseteq (x)_{\mathcal{P}\gamma}$, *i.e.*, e_x is a terminal link, Because no isolated vertex in any premap, from Lemma 3.3, $M - e_x = (\mathcal{X} - Kx, \mathcal{P}_{-x})$ is a map. In what follows, this case is not considered again.

Necessity. Suppose $M - e_x$ is a map, but e_x is a harmonic link of M. Because $\mathcal{P}\gamma^{-1}x$ and $\mathcal{P}\gamma^{-1}\gamma x \neq x(\langle x, \gamma x\rangle \nsubseteq (x)_{\mathcal{P}\gamma})$ for group $\Psi_{J'}$, $J' = \{\mathcal{P}_{-x}, \alpha, \beta\}$, are not transitive on the set $\mathcal{X} - Kx$, $M - e_x$ is not a map. This is a contradiction to the assumption.

Sufficiency. Because M is a map, $\mathcal{P}_{-x}$ is basic. From Theorem 2.3, $\mathcal{P}_{-x}$ satisfies the conjugate axiom. Then, based on Table 3.1, two cases should be discussed for the transitivity.

(i) When e_x is a single edge(including single loops!) or singular link of M. Because e_x is not a cut-edge of its under graph $G(M)$,

$G(M - e_x)$ is connected. From Theorem 2.6, $\mathcal{P}_{-x}$ for group $\Psi_{J'}$ is transitive on the set $\mathcal{X} - Kx$. Thus, $M - e_x$ is a map.

(ii) When e_x is a double loop of M. Because

$$((\mathcal{P}\gamma)^{-1}x)_{\mathcal{P}_{-x}} = ((\mathcal{P}\gamma)^{-1}x, \mathcal{P}\alpha x, \cdots, \\ \beta\mathcal{P}\beta x, \mathcal{P}\gamma x, \cdots, (\mathcal{P}\gamma)^{-2})\alpha x),$$

we have $\mathcal{P}x = \beta(\mathcal{P}\gamma)x$ with $(\mathcal{P}\gamma)^{-1}\alpha x$ and $\mathcal{P}\beta x = \beta(\beta\mathcal{P}\beta x)$ are transitive in $M - e_x$. Hence, $M - e_x$ is a map as well. □

From the proof of the theorem, a much fundamental conclusion is soon deduced.

Corollary 3.1 In a map M, an edge e_x is a segmentation edge if, and only if, e_x is a harmonic link except for terminal links. And, e_x is a harmonic link if, and only if, it is a cut-edge of graph $G(M)$.

Proof To prove the first statement.

Necessity. Because e_x is a segmentation edge, $G(M - e_x)$ is not connected. e_x is only a link. On the basis of Table 3.1, e_x is also a double edge. And, because e_x is not singular, e_x is only harmonic. Clearly, a terminal link is not a segmentation edge in its own right.

Sufficiency. Because e_x is not a terminal link, $\mathcal{P}x$ is distinct form x and γx is distinct from $\mathcal{P}\gamma x$. And, because e_x is a harmonic link, $\mathcal{P}x$ and $\mathcal{P}\gamma x$ are not transitive in $M - e_x$. Thus, e_x is a segmentation edge.

To prove the second statement. Because it can be shown that e_x is a terminal link of M if, and only if, e_x is an articulate edge of graph $G(M)$, a cut-edge as well. This statement is deduced from the first statement. □

Let $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$ be a pregraph and $e_x = Kx = \{x, \alpha x, \beta x, \gamma x\}$, $x \in X$, be an edge. The *contraction* of e_x from M, denoted by

$$M \bullet e_x = (\mathcal{X}_{\alpha,\beta}(X) - Kx, \mathcal{P}_{\bullet x}),$$

is defined to be the dual of $M^* - e_x^*$ where $e_x^* = \{x, \beta x, \alpha x, \gamma x\}$, the corresponding edge of e_x in the dual M^* of M. In other words, $\mathcal{P}_{\bullet x} = \mathcal{P}^*_{-x}\gamma$.

Lemma 3.4 $\mathcal{P}_{\bullet x}$ is determined by the following (i–iii):

(i) When e_x is a link. For $y \in \mathcal{X}_{\alpha,\beta}(X) - Kx$,

$$\mathcal{P}_{\bullet x}y = \begin{cases} \mathcal{P}\gamma x(\text{and } \alpha\mathcal{P}^{-1}x), \text{ if } y = \mathcal{P}^{-1}x(\text{and } \alpha\mathcal{P}\gamma x); \\ \mathcal{P}x(\text{and } \alpha\mathcal{P}^{-1}\gamma x), \text{ if } y = \mathcal{P}^{-1}\gamma x(\text{and } \alpha\mathcal{P}x); \\ \mathcal{P}y, \text{ otherwise,} \end{cases} \tag{3.5}$$

shown as in Fig.3.3(a)$\Longrightarrow$(b).

(ii) When e_x is a harmonic loop. For $y \in \mathcal{X}_{\alpha,\beta}(X) - Kx$,

$$\mathcal{P}_{\bullet x}y = \begin{cases} \mathcal{P}\gamma x(\text{and } \alpha\mathcal{P}^{-1}x), \text{ if } y = \mathcal{P}^{-1}x(\text{and } \alpha\mathcal{P}\gamma x); \\ \mathcal{P}x(\text{and } \alpha\mathcal{P}^{-1}\gamma x), \text{ if } y = \mathcal{P}^{-1}\gamma x(\text{and } \alpha\mathcal{P}x); \\ \mathcal{P}y, \text{ otherwise,} \end{cases} \tag{3.6}$$

shown as in Fig.3.3(c)$\Longrightarrow$(d).

(iii) When e_x is a singular , or double loop. For $y \in \mathcal{X}_{\alpha,\beta}(X) - Kx$,

$$\mathcal{P}_{\bullet x}y = \begin{cases} \alpha\mathcal{P}^{-1}\beta x(\text{and } \alpha\mathcal{P}^{-1}x), \\ \quad \text{if } y = \mathcal{P}^{-1}x(\text{and } \mathcal{P}^{-1}\beta x); \\ \mathcal{P}\beta x(\text{and } \mathcal{P}x), \\ \quad \text{if } y = \alpha\mathcal{P}x(\text{and } \alpha\mathcal{P}\beta x); \\ \mathcal{P}y, \text{ otherwise,} \end{cases} \tag{3.7}$$

shown as in Fig.3.3(e)$\Longrightarrow$(f).

Proof (i) When e_x is a link. In the dual M^* of M, from the duality,

$$(x)_{\mathcal{P}^*} = (x, \mathcal{P}\gamma x, (\mathcal{P}\gamma)^2 x, \cdots, (\mathcal{P}\gamma)^{-1}x)$$

and

$$(\gamma x)_{\mathcal{P}^*} = (\gamma x, \mathcal{P}x, (\mathcal{P}\gamma)^2\gamma x, \cdots, (\mathcal{P}\gamma)^{-1}\gamma x),$$

or

$$(x)_{\mathcal{P}^*} = (x, \mathcal{P}\gamma x, \cdots, (\mathcal{P}\gamma)^{-1}\gamma x, \gamma x, \mathcal{P}x, \cdots, (\mathcal{P}\gamma)^{-1}x),$$

and hence $\mathcal{P}^*_{-x}$ is only different from $\mathcal{P}^* = \mathcal{P}\gamma$ at vertices

$$(\mathcal{P}\gamma x)_{\mathcal{P}^*_{-x}} = (\mathcal{P}\gamma x, (\mathcal{P}\gamma)^2 x, \cdots, (\mathcal{P}\gamma)^{-1}x)$$

and

$$(\mathcal{P}x)_{\mathcal{P}^*_{-x}} = (\mathcal{P}x, (\mathcal{P}\gamma)^2\gamma x, \cdots, (\mathcal{P}\gamma)^{-1}\gamma x)$$

or at vertex

$$(\mathcal{P}x)_{\mathcal{P}^*_{-x}} = (\mathcal{P}\gamma x, \cdots, (\mathcal{P}\gamma)^{-1}\gamma x, \mathcal{P}x, \cdots, (\mathcal{P}\gamma)^{-1}x)$$

with their conjugations according as e_x is single, or double. By considering $\mathcal{P}_{\bullet x} = \mathcal{P}^*_{-x}\gamma$,

$$\begin{aligned}\mathcal{P}_{\bullet x}(y) &= \mathcal{P}_{\bullet x}(\mathcal{P}^{-1}\gamma x) = \mathcal{P}^*_{-x}\gamma(\mathcal{P}^{-1}\gamma x)\\ &= \mathcal{P}^*_{-x}(\mathcal{P}\gamma^{-1}\gamma x) = \mathcal{P}x\end{aligned}$$

for $y = \mathcal{P}^{-1}\gamma x$ and

$$\begin{aligned}\mathcal{P}_{\bullet x}(y) &= \mathcal{P}_{\bullet x}(\mathcal{P}^{-1}x) = \mathcal{P}^*_{-x}\gamma(\mathcal{P}^{-1}x)\\ &= \mathcal{P}^*_{-x}(\mathcal{P}\gamma)^{-1}x = \mathcal{P}\gamma x\end{aligned}$$

for $y = \mathcal{P}^{-1}x$. From the conjugate axiom, the cases for $y = \alpha\mathcal{P}^{-1}x$ and $y = \alpha(\mathcal{P}\gamma x)$ in the parentheses of (3.5) are also obtained. Then, for other y,

$$\mathcal{P}_{\bullet x}(y) = \mathcal{P}^*_{-x}\gamma y = (\mathcal{P}\gamma)\gamma y = \mathcal{P}y$$

in the both cases. Therefore, (3.5) is true.

(ii) When e_x is a harmonic loop. In a similar way to (3.5) for e_x single, (3.6) is also obtained.

iii) When e_x is a singular, or double loop. In the dual M^* of M,

$$(x)_{\mathcal{P}^*} = (x, \mathcal{P}\gamma x, (\mathcal{P}\gamma)^2 x, \cdots, (\mathcal{P}\gamma)^{-1}x)$$

and

$$(\alpha x)_{\mathcal{P}^*} = (\alpha x, \mathcal{P}\gamma\alpha x, (\mathcal{P}\gamma)^2\alpha x, \cdots, (\mathcal{P}\gamma)^{-1}\alpha x),$$

or

$$(x)_{\mathcal{P}^*} = (x, \mathcal{P}\gamma x, \cdots, (\mathcal{P}\gamma)^{-1}\alpha x, \alpha x, \mathcal{P}\gamma\alpha x, \cdots, (\mathcal{P}\gamma)^{-1}x),$$

and hence

$$(x)_{\mathcal{P}^*_{-x}} = (\mathcal{P}\gamma x, (\mathcal{P}\gamma)^2 x, \cdots, (\mathcal{P}\gamma)^{-1}x)$$

and

$$(\alpha x)_{\mathcal{P}^*_{-x}} = (\mathcal{P}\gamma\alpha x, (\mathcal{P}\gamma)^2\alpha x, \cdots, (\mathcal{P}\gamma)^{-1}\alpha x)$$

or

$$(x)_{\mathcal{P}^*_{-x}} = (\mathcal{P}\gamma x, \cdots, (\mathcal{P}\gamma)^{-1}\alpha x, \mathcal{P}\gamma\alpha x, \cdots, (\mathcal{P}\gamma)^{-1}x)$$

with their conjugations according as e_x is singular, or double. By considering $\mathcal{P}_{\bullet x} = \mathcal{P}^*_{-x}\gamma$,

$$\begin{aligned}\mathcal{P}_{\bullet x}(y) &= \mathcal{P}_{\bullet x}(\alpha\mathcal{P}x) = \mathcal{P}^*_{-x}\gamma(\alpha\mathcal{P}x)\\ &= \mathcal{P}^*_{-x}\gamma(\mathcal{P}^{-1}\alpha x) = \mathcal{P}^*_{-x}((\mathcal{P}\gamma)^{-1}\alpha x)\\ &= \mathcal{P}\gamma\alpha x = \mathcal{P}\beta x\end{aligned}$$

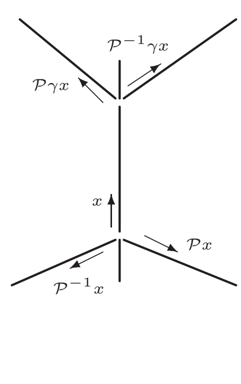
for $y = \alpha\mathcal{P}x$ and

$$\begin{aligned}\mathcal{P}_{\bullet x}(y) &= \mathcal{P}_{\bullet x}(\mathcal{P}^{-1}x) = \mathcal{P}^*_{-x}\gamma(\mathcal{P}^{-1}x)\\ &= \mathcal{P}^*_{-x}((\mathcal{P}\gamma)^{-1}x) = \mathcal{P}\gamma x\\ &= \mathcal{P}\alpha\beta x = \alpha\mathcal{P}^{-1}\beta x\end{aligned}$$

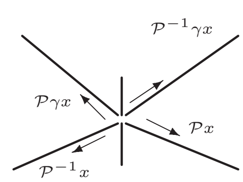
for $y = \mathcal{P}^{-1}x$. From the conjugate axiom, the cases for $y = \mathcal{P}\beta x$ and $\alpha\mathcal{P}\beta x$ are also obtained. Then, for all other y,

$$\mathcal{P}_{\bullet x}(y) = \mathcal{P}^*_{-x}\gamma y = (\mathcal{P}\gamma)\gamma y = \mathcal{P}y.$$

This is (3.7). □



(a)



(b)

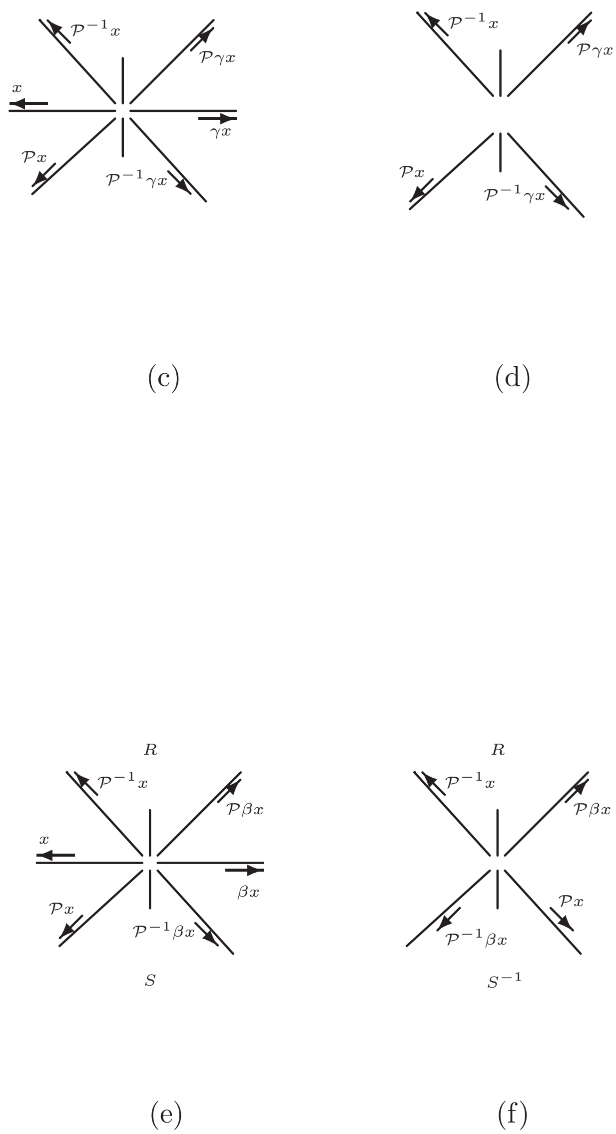


Fig.3.3 Contraction of an edge

From Lemma 3.4, it is seen that in the constriction of edge e_x on a premap only if e_x is not a selfloop, two vertices $(x)_{\mathcal{P}}$ and $(\gamma x)_{\mathcal{P}}$ are composed of one vertex

$$(\mathcal{P}^{-1}x)_{\mathcal{P}_{\bullet x}} = (\mathcal{P}^{-1}x, \mathcal{P}\gamma x, \cdots, \mathcal{P}^{-1}\gamma x, \mathcal{P}x, \cdots, \mathcal{P}^{-2}x)$$

(Fig.3.3(a)⇒(b)); if e_x is a harmonic loop, vertex $(x)_{\mathcal{P}}$ is divided into two vertices

$$(\mathcal{P}^{-1}x)_{\mathcal{P}_{\bullet x}} = (\mathcal{P}^{-1}x, \mathcal{P}\gamma x, \cdots, \mathcal{P}^{-2}x)$$

and

$$(\mathcal{P}^{-1}\gamma x)_{\mathcal{P}_{\bullet x}} = (\mathcal{P}^{-1}\gamma x, \mathcal{P}x, \cdots, \mathcal{P}^{-2}\gamma x)$$

(Fig.3.2(c)⇒(d)); and if e_x is a singular, or double loop, vertex $(x)_{\mathcal{P}}$ becomes vertex

$$(\mathcal{P}x)_{\mathcal{P}_{\bullet x}} = (\mathcal{P}x, \cdots, \mathcal{P}^{-1}\beta x, \alpha\mathcal{P}^{-1}x, \cdots, \alpha\mathcal{P}\beta x)$$

(Fig.3.3(e)⇒(f)).

Lemma 3.5 For a premap M, $M \bullet e_x$ is always a premap. And, the number of transitive blocks in $M \bullet e_x$ is not less than that in M.

Proof From Lemma 3.3 and the duality, the first statement is true. Because any nontransitive pair of elements in M is never transitive in $M \bullet e_x$ from Lemma 3.4, the second statement is true. $\square$

If a harmonic loop e_x has $(x)_{\mathcal{P}\gamma} = (x)$, or $(\gamma x)_{\mathcal{P}\gamma} = (\gamma x)$, then it is called a *terminal loop*. If the two elements of a co-semiedge appear in a vertex in succession, then the edge is called a *twist loop*.

Lemma 3.6 For an edge e_x of a map M, e_x is a terminal loop if, and only if, e_x^* is an terminal link in M^*. And, e_x is a twist loop if, and only if, e_x^* is a twist loop.

Proof A direct result deduced from the duality. $\square$

Theorem 3.5 For an edge e_x of a map $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$, $M \bullet e_x$ is a map if, and only if, e_x is not a harmonic loop but terminal loop.

Proof Because for a terminal loop e_x, $M \bullet e_x$ is always a map. In what follows, this case is excluded.

Necessity. Suppose $M \bullet e_x$ is a map but e_x is a harmonic loop. Since e_x^* is a harmonic link in M^*(Table 3.1), from Theorem 3.4 and Lemma 3.1, $\mathcal{P}^{-1}x$ and $\mathcal{P}x$ are, respectively, belonging to two distinct transitive blocks of M. From Lemma 3.4(ii), $M \bullet e_x$ has two transitive blocks. This contradicts to that $M \bullet e_x$ is a map.

Sufficiency. Since e_x is not a harmonic loop, only two cases should be considered as e_x is not a loop or e_x is a singular loop. For the former, in spite of a single or double edge, from Lemma 3.4(i), $M \bullet e_x$ is a map. For the latter, from Lemma 3.4(iii), $M \bullet e_x = M - e_x$ is also a map.

Therefore, the theorem is done. □

If a loop e_x has that $\mathcal{P}^{-1}x$ and $\mathcal{P}x$ are in distinct noncuttable blocks, then it is called a *shearing loop* . From Theorem 3.5, all shearing loops are harmonic. However, the converse case is unnecessarily true.

Corollary 3.2 In a map M, an edge e_x is a shearing loop if, and only if, e_x^* is a harmonic, but not terminal loop in M^*.

Proof A direct result of Theorem 3.5. □

Theorem 3.6 The dual of premap $M - e_x$ is the premap $M^* \bullet e_x^*$, where M^* is the dual of M and e_x^* in M^* is the corresponding edge of e_x in M.

Proof(1) Because $M^* \bullet e_x^*$ is the dual of $(M^*)^* - e_x^{**} = M - e_x$, by the symmetry of the duality the theorem holds. □

However, if the contraction of e_x on M is defined by (3.5–7), then the theorem can also be proved.

Proof(2) Based on Table 3.1, four cases should be discussed.

(i) In M, e_x is a single link, and hence e_x^* is a single link in M^*. Since

$$(\mathcal{P}\gamma x)_{\mathcal{P}_{-x}\gamma} = (\mathcal{P}\gamma x)_{\mathcal{P}^*_{\bullet x}}, \ (\mathcal{P}x)_{\mathcal{P}_{-x}} = (\mathcal{P}x)_{\mathcal{P}^*_{\bullet x}\gamma}$$

and $(\mathcal{P}\gamma x)_{\mathcal{P}_{-x}} = (\mathcal{P}\gamma x)_{\mathcal{P}^*_{\bullet x}\gamma}$,

$$(M - e_x)^* = M^* \bullet e_x^*.$$

(ii) In M, e_x is a harmonic link, and hence e_x^* is a harmonic loop in M^*(Dually, in M, e_x is a harmonic loop, and hence e_x^* is a harmonic link in M^*). Now, $(x)_{\mathcal{P}\gamma} = (x)_{\mathcal{P}^*}$. According as e_x is a terminal link or not, a transitive block of M becomes one or two transitive blocks in $M - e_x$. Meanwhile, According as e_x^* is a terminal loop or not, a transitive block of M^* becomes one or two transitive blocks of $M^* \bullet e_x^*$. By considering the changes in vertices and faces, $(M - e_x)^* = M^* \bullet e_x^*$ is found.

(iii) In M, e_x is a singular link, and hence e_x^* is a singular loop in M^*(Dually, In M, e_x is a singular loop, and hence e_x^* is a singular link in M^*). Since

$$(\mathcal{P}^{-1}x)_{\mathcal{P}_{-x}} = (\mathcal{P}^{-1}x)_{\mathcal{P}^*_{\bullet x}\gamma}$$

and

$$(\mathcal{P}^{-1}\gamma x)_{\mathcal{P}_{-x}} = (\mathcal{P}^{-1}\gamma x)_{\mathcal{P}^*_{\bullet x}\gamma},$$

in view of $(\mathcal{P}\gamma x)_{\mathcal{P}_{-x}\gamma} = (\mathcal{P}\gamma x)_{\mathcal{P}^*_{\bullet x}}$ we have $(M - e_x)^* = M^* \bullet e_x^*$.

(iv) In M, e_x is a double loop, and hence e_x^* is a double loop in M^*. Since

$$(\mathcal{P}x)_{\mathcal{P}_{-x}} = (\mathcal{P}x)_{\mathcal{P}^*_{\bullet x}\gamma}$$

and

$$(\mathcal{P}x)_{\mathcal{P}_{-x}\gamma} = (\mathcal{P}x)_{\mathcal{P}^*_{\bullet x}},$$

we have $(M - e_x)^* = M^* \bullet e_x^*$. □

Corollary 3.3 In a map, an edge is a harmonic link if, and only if, the corresponding edge in its dual is a harmonic loop. And, an edge is a segmentation edge if, and only if, the corresponding edge in its dual is a shearing loop.

Proof A direct result of Theorem 3.6. □

Example 3.2 Map $M = (Kx + Ky + Kz, \mathcal{P})$ where

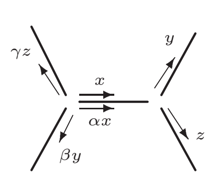
$$\mathcal{P} = (x, \beta y, \gamma z)(y, z, \gamma x)$$

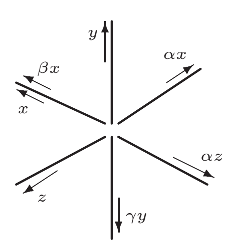
and its dual $M^* = (K^*x + K^*y + K^*z, \mathcal{P}^*)$ where

$$\mathcal{P}^* = \mathcal{P}\gamma = (x, y, \alpha x, \alpha z, \gamma y, z),$$

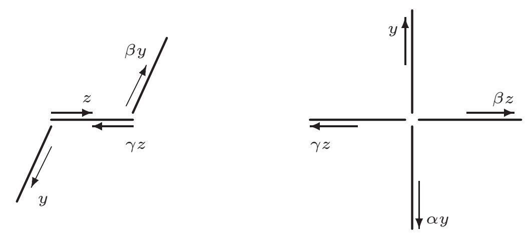
are, respectively, shown in Fig.3.4(a) and (b). Here, $K = \{1, \alpha, \beta, \gamma\}$ and $K^* = \{1, \beta, \alpha, \gamma\}$ are used to distinguish α and β.

Map $M - e_x = (Ky + Kz, \mathcal{P}_{-x})$ where $\mathcal{P}_{-x} = (\beta y, \gamma z)(y, z)$ and its dual $(M - e_x)^* = (K^*y + K^*z, (\mathcal{P}_{-x})^*)$, where $(\mathcal{P}_{-x})^* = \mathcal{P}_{-x}\gamma = (y, \beta z, \alpha y, \gamma z)$, are, respectively, shown in Fig.3.4(c) and (d). It is easily seen that $(M - e_x)^* = M^* \bullet e_x^*$.





(a) M (b) M^*



(c) $M - e_x$ (d) $(M - e_x)^*$

Fig.3.4 Duality between deletion and contraction

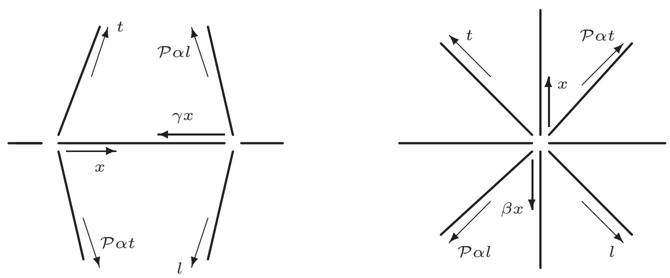
III.3 Addition of an edge

Let $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ be a premap, $e_x = Kx = \{x, \alpha x, \beta x, \gamma x\}$, and $x \notin \mathcal{X}_{\alpha,\beta}$. Write as

$$M + e_x = (\mathcal{X}_{\alpha,\beta} + Kx, \mathcal{P}_{+x}),$$

where $\mathcal{P}_{+x}$ is determined from $\mathcal{P}$ in the following manner. For any $y \in \mathcal{X}_{\alpha,\beta}$ and two given angles $\langle l, \mathcal{P}\alpha l\rangle$ and $\langle t, \mathcal{P}\alpha t\rangle$, if l and t are not at the same vertex, or at the same vertex and e_x as a harmonic loop(assume $t \in (l)_{\mathcal{P}}$ without loss of generality), then

$$\mathcal{P}_{+x}y = \begin{cases} t(\text{and } \alpha x), \text{ if } y = x(\text{and } \alpha t); \\ \mathcal{P}\alpha t(\text{and } x), \text{ if } y = \alpha x(\text{and } \alpha\mathcal{P}\alpha t); \\ l(\text{and } \beta x), \text{ if } y = \gamma x(\text{and } \alpha l); \\ \mathcal{P}\alpha l(\text{and } \gamma x), \text{ if } y = \beta x(\text{and } \alpha\mathcal{P}\alpha l); \\ \mathcal{P}y, \text{ otherwise}, \end{cases} \tag{3.8}$$



(a) For (3.8) (b) For (3.9)

Fig.3.5 Appending an edge

shown in Fig.3.5(a), otherwise, *i.e.*, e_x is a double, or singular loop (assume $t \in (l)_{\mathcal{P}}$ without loss of generality),

$$\mathcal{P}_{+x}y = \begin{cases} t(\text{and } \alpha x), \text{ if } \ y = x(\text{and } \alpha t); \\ \mathcal{P}\alpha t(\text{and } x), \text{ if } \ y = \alpha x(\text{and}\alpha\mathcal{P}\alpha t); \\ l(\text{and}\beta x), \text{ if } \ y = \gamma x(\text{and } \alpha l); \\ \mathcal{P}\alpha l(\text{and } \beta x), \text{ if } \ y = \gamma x(\text{and } \alpha\mathcal{P}\alpha l); \\ \mathcal{P}y, \text{ otherwise,} \end{cases} \tag{3.9}$$

as shown in Fig.3.5(b).

Such a transformation from M into $M + e_x$ is called *appending an edge* e_x.

Lemma 3.7 For a premap $M = (\mathcal{X}, \mathcal{P})$, $M + e_x = (\mathcal{X} + Kx, \mathcal{P}_{+x})$ is also a premap. And, the number of transitive blocks in $M + e_x$ is not greater than that in M.

Proof From (3.8) and (3.9), $\mathcal{P}_{+x}$ is basic. In virtue of Theorem 2.3, it suffices to show that the orbits of $\mathcal{P}_{+x}$ are partitioned into conjugate pairs for the conjugate axiom. In fact, if l and t are at distinct vertices, then $\mathcal{P}_{+x}$ is obtained from $\mathcal{P}$ in replacing two vertices

$(t)_{\mathcal{P}}$ and $(l)_{\mathcal{P}}$ by respective

$$(t)_{\mathcal{P}_{+x}} = (t, \mathcal{P}t, \cdots, \alpha\mathcal{P}\alpha t, x)$$

and

$$(l)_{\mathcal{P}_{+x}} = (l, \mathcal{P}l, \cdots, \alpha\mathcal{P}\alpha l, \gamma x),$$

or βx is substituted for γx. If l and t are at the same vertex, then $\mathcal{P}_{+x}$ is obtained from $\mathcal{P}$ in replacing the vertex $(l)_{\mathcal{P}}$ by

$$(l)_{\mathcal{P}_{+x}} = (l, \mathcal{P}l, \cdots, \alpha\mathcal{P}\alpha t, x, t, \mathcal{P}t, \cdots, \alpha\mathcal{P}\alpha l, \gamma x)$$

or

$$(l)_{\mathcal{P}_{+x}} = (l, \mathcal{P}l, \cdots, \alpha\mathcal{P}\alpha t, x, t, \mathcal{P}t, \cdots, \alpha\mathcal{P}\alpha l, \beta x)$$

according as e_x is a harmonic loop or not. This shows that the orbits of $\mathcal{P}_{+x}$ are partitioned into conjugate pairs for α. □

Note 3.2 On the degenerate case $t = l$, if e_x is a harmonic loop, then

$$(l)_{\mathcal{P}_{+x}} = (l, \mathcal{P}l, \cdots, \alpha\mathcal{P}\alpha l, \gamma x, x);$$

otherwise, *i.e.*, e_x is a twist loop,

$$(l)_{\mathcal{P}_{+x}} = (l, \mathcal{P}l, \cdots, \alpha\mathcal{P}\alpha l, \beta x, x).$$

Theorem 3.7 For a premap, not a map, $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, the number of transitive blocks of $M + e_x$ is less than that of M if, and only if, e_x ia a segmentation edge. If M is a map, then $M + e_x$ is also a map.

Proof Since the number of components of graph $G(M + e_x)$ is less than that of $G(M)$ if, and only if, e_x is a cut-edge which is not articulate, the first statement is deduced from Corollary 3.1.

Because the transitivity between two elements in the ground set of M under appending an edge is unchanged, the second statement is valid. □

Note 3.3 Let $M' = (\mathcal{X} + Kx, \mathcal{P}') = M + e_x$ for $M = (\mathcal{X}, \mathcal{P})$. Because $\mathcal{P}'_{-x} = \mathcal{P}$, M is obtained by the deletion of the edge e_x from

M', *i.e.*, $M = M' - e_x$. Therefore, the operation of appending an edge on a premap is the inverse of the corresponding edge deletion.

Now, another operation for increasing by an edge on a premap is considered. This is the splitting an edge seen as the inverse of edge contraction.

Let $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ be a premap. Suppose $\langle l, \mathcal{P}\alpha l\rangle$ and $\langle t, \mathcal{P}\alpha t\rangle$ are two angles. For $x \notin \mathcal{X}_{\alpha,\beta}$, let $M \circ e_x = (\mathcal{X}_{\alpha,\beta} + Kx,\ \mathcal{P}_{\circ x})$, where $\mathcal{P}_{\circ x}$ is determined by $\mathcal{P}$ in the following manner. The transformation from M into $M \circ e_x$ is called *splitting* an edge e_x and e_x, the *splitting edge* of M.

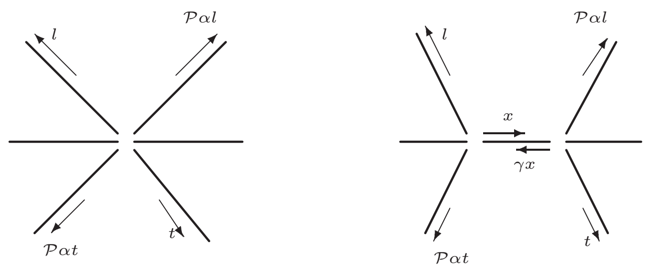
Lemma 3.8 Let $l \in \{t\}_{\mathcal{P}} \cup \{\alpha t\}_{\mathcal{P}}$. If $l \notin (t)_{\mathcal{P}\gamma} \cup (\beta t)_{\mathcal{P}\gamma}$, then

$$\mathcal{P}_{\circ x} y = \begin{cases} \mathcal{P}\alpha t(\text{or } \alpha x), \text{ if } y = x(\text{or } \alpha\mathcal{P}\alpha t); \\ l(\text{or } x), \text{ if } y = \alpha x(\text{or } \alpha l), \\ \mathcal{P}\alpha l(\text{or } \beta x), \text{ if } y = \gamma x(\text{or } \alpha\mathcal{P}\alpha l); \\ t(\text{or } \gamma x), \text{ if } y = \beta x(\text{or } \alpha t); \\ \mathcal{P}y, \text{ otherwise.} \end{cases} \tag{3.10}$$

The edge e_x is a single link as shown in Fig.3.6; Otherwise, *i.e.*, $l \in (t)_{\mathcal{P}\gamma} \cup (\beta t)_{\mathcal{P}\gamma}$, then

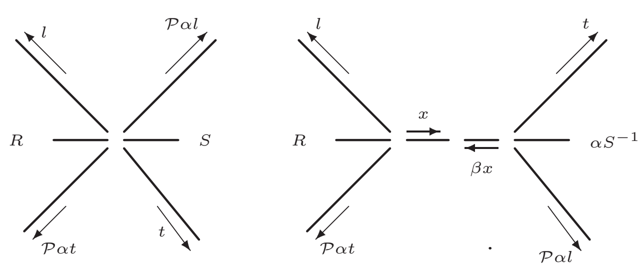
$$\mathcal{P}_{\circ x} y = \begin{cases} \mathcal{P}\alpha t(\text{or } \alpha x), \text{ if } y = x(\text{or } \alpha\mathcal{P}\alpha t); \\ l(\text{or } x), \text{ if } y = \alpha x(\text{or } \alpha l); \\ t(\text{or } \beta x), \text{ if } y = \beta x(\text{or } \alpha t); \\ \mathcal{P}\alpha l(\text{or } \beta x), \text{ if } y = \gamma x(\text{or } \alpha\mathcal{P}\alpha l); \\ \mathcal{P}y, \text{ otherwise,} \end{cases} \tag{3.11}$$

or γx replaced by βx to attain, respectively, e_x as a singular or harmonic link shown in Fig.3.7.



(a) M (b) $M \circ e_x$

Fig.3.6 $l \notin (t)_{\mathcal{P}\gamma} \cup (\beta t)_{\mathcal{P}\gamma}$



(a) M (b) $M \circ e_x$

Fig.3.7 $l \in (t)_{\mathcal{P}\gamma} \cup (\beta t)_{\mathcal{P}\gamma}$

Proof Since $l \in \{t\}_{\mathcal{P}} \cup \{\alpha t\}_{\mathcal{P}}$, l and t are at the same vertex. Thus, e_x is a link. If $l \notin (t)_{\mathcal{P}\gamma} \cup (\beta t)_{\mathcal{P}\gamma}$, *i.e.*, l is in a face different from that t is in, or in other words, e_x is single, then by the reason as $(\mathcal{P}_{\circ x})_{\bullet x}$ is different from only the vertex

$$(\mathcal{P}_{\circ x}x)_{(\mathcal{P}_{\circ x})_{\bullet x}} = (\mathcal{P}\alpha t, \cdots, \alpha l, \mathcal{P}\alpha l, \cdots, \alpha t)$$

where $\mathcal{P}_{\circ x}x = \mathcal{P}\alpha t$ shown in Fig.3.6, from Lemma 3.4(i), $(\mathcal{P}_{\circ x})_{\bullet x} = \mathcal{P}$. Otherwise, according as e_x is singular or harmonic, $(\mathcal{P}_{\circ x})_{\bullet x}$ is different from only the vertex

$$(\mathcal{P}_{\circ x}x)_{(\mathcal{P}_{\circ x})_{\bullet x}} = (\mathcal{P}\alpha t, \cdots, \alpha l, \mathcal{P}\alpha l, \cdots, \alpha t)$$

or

$$(\mathcal{P}\alpha t, \cdots, \alpha l, t, \cdots, \alpha\mathcal{P}\alpha l)$$

shown in Fig.3.7. From Lemma 3.4(i) again, $(\mathcal{P}_{\circ x})_{\bullet x} = \mathcal{P}$. □

Lemma 3.9 Let $l \notin \{t\}_{\mathcal{P}} \cup \{\alpha t\}_{\mathcal{P}}$. If t and l are not transitive on M, then

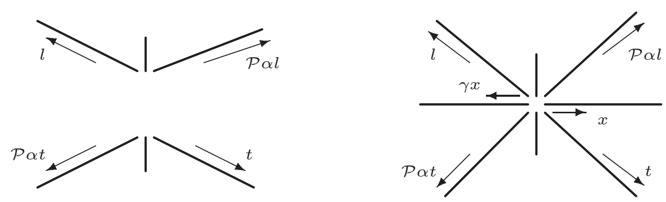
$$\mathcal{P}_{\circ x}y = \begin{cases} \mathcal{P}\alpha l(\text{or } \alpha x), & \text{if } y = x(\text{or } \alpha\mathcal{P}\alpha l); \\ t(\text{or } x), & \text{if } y = \alpha x(\text{or } \alpha t); \\ \mathcal{P}\alpha t(\text{or } \beta x), & \text{if } y = \gamma x(\text{or } \alpha\mathcal{P}\alpha t); \\ l(\text{or } \gamma x), & \text{if } y = \beta x(\text{or } \alpha l); \\ \mathcal{P}y, & \text{otherwise} \end{cases} \quad (3.12)$$

as shown in Fig.3.8 where e_x is a harmonic loop.

Otherwise, *i.e.*, t and l are transitive on M, then

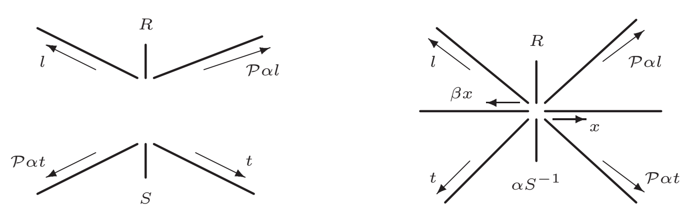
$$\mathcal{P}_{\circ x}y = \begin{cases} \mathcal{P}\alpha l(\text{or } \alpha x), & \text{if } y = x(\text{or } \alpha\mathcal{P}\alpha l); \\ \mathcal{P}\alpha t(\text{or } x), & \text{if } y = \alpha x(\text{or } \alpha\mathcal{P}\alpha t); \\ l(\text{or } \beta x), & \text{if } y = \gamma x(\text{or } \alpha l); \\ t(\text{or } \gamma x), & \text{if } y = \beta x(\text{or } \alpha t); \\ \mathcal{P}y, & \text{otherwise,} \end{cases} \quad (3.13)$$

as shown in Fig.3.9 where e_x is a singular, or harmonic loop according as it is incident with two faces, or one face.



(a) M (b) $M \circ e_x$

Fig.3.8 t and l nontransitive



(a) M (b) $M \circ e_x$

Fig.3.9 t and l transitive

Proof Since $l \in \{t\}_{\mathcal{P}} \cup \{\alpha t\}_{\mathcal{P}}$, l and t are at the same vertex. Thus, e_x is a link. If $l \notin (t)_{\mathcal{P}\gamma} \cup (\beta t)_{\mathcal{P}\gamma}$, *i.e.*, l is in a face different from that t is in, or in other words, e_x is single, then by the reason as

$(\mathcal{P}_{\circ x})_{\bullet x}$ is different from only the vertex

$$(\mathcal{P}_{\circ x}x)_{(\mathcal{P}_{\circ x})_{\bullet x}} = (\mathcal{P}\alpha t, \cdots, \alpha l, \mathcal{P}\alpha l, \cdots, \alpha t)$$

where $\mathcal{P}_{\circ x}x = \mathcal{P}\alpha t$ shown in Fig.3.6, from Lemma 3.4(i), $(\mathcal{P}_{\circ x})_{\bullet x} = \mathcal{P}$. Otherwise, according as e_x is singular or harmonic, $(\mathcal{P}_{\circ x})_{\bullet x}$ is different from only the vertex

$$(\mathcal{P}_{\circ x}x)_{(\mathcal{P}_{\circ x})_{\bullet x}} = (\mathcal{P}\alpha t, \cdots, \alpha l, \mathcal{P}\alpha l, \cdots, \alpha t)$$

or

$$(\mathcal{P}\alpha t, \cdots, \alpha l, t, \cdots, \alpha\mathcal{P}\alpha l)$$

shown in Fig.3.7. From Lemma 3.4(i) again, $(\mathcal{P}_{\circ x})_{\bullet x} = \mathcal{P}$. □

Lemma 3.10 If $M = (\mathcal{X}, \mathcal{P})$ is a premap, then for any $x \in \mathcal{X}$, $M \circ e_x = (\mathcal{X} + Kx, \mathcal{P}_{\circ x})$ is also a premap. And, $M \circ e_x$ has the number of its transitive blocks not greater than M does.

Proof From Lemmas 3.8–9, permutation $\mathcal{P}_{\circ x}$ is basic and partitioned into conjugate pairs for α. Then by Theorem 2.3, $M \circ x$ is also a premap. This is the first statement. Because splitting an edge does not changing the transitivity of any pair of elements in $\mathcal{X}$, the second statement holds. □

Lemma 3.11 Edge e_x is a harmonic loop on $M \circ e_x$ if, and only if, $\mathcal{P}_{\circ x}x$ and $\mathcal{P}_{\circ x}\gamma x$ are not transitive on M.

Proof Necessity. Because the splitting edge e_x is a harmonic loop on M, it is in the case (3.12). As shown in Fig. 3.8, $\mathcal{P}_{\circ x}\beta x(= l)$ and $\mathcal{P}_{\circ x}\alpha x(= t)$ are not transitive on M. Hence, by the symmetry among elements in Kx, $\mathcal{P}_{\circ x}x$ and $\mathcal{P}_{\circ x}\gamma x$ are not transitive on M either.

Sufficiency. Because $\mathcal{P}_{\circ x}\beta x(= l)$ and $\mathcal{P}_{\circ x}\alpha x(= t)$ are not transitive on M, only $l \notin \{t\}_{\mathcal{P}} \cup \{\alpha t\}_{\mathcal{P}}$ is possible. This is the case for (3.12). Thus, e_x is a harmonic loop. □

Theorem 3.8 For a premap $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ not a map, the number of transitive blocks in $M \circ e_x$ is less than that in M if, and only if, e_x is a harmonic loop. If M is a map, then $M \circ e_x$ is also a map.

Proof From Lemma 3.11, the number of transitive blocks in $M \circ e_x$ is less than that in M if, and only if, e_x is a segmentation edge in $M + e_x$. This is the first statement. Because splitting an edge in a map does not changing the transitivity, the second statement is obtained. □

Lemma 3.12 Edge $e_x = \{x, \alpha x, \beta x, \gamma x\}$ is appended in premap M if, and only if, edge $e_x^* = \{x, \beta x, \alpha x, \gamma x\}$ is split to in premap M^*.

Proof Necessity. (1) If e_x is a single edge of $M + e_x$, *i.e.*, $\gamma x \notin (x)_{\mathcal{P}_{+x}\gamma} \cup (\beta x)_{\mathcal{P}_{+x}\gamma}$, then $(x)_{\mathcal{P}^*_{+x}}$ and $(\gamma x)_{\mathcal{P}^*_{+x}}$ are two vertices on $(M + e_x)^*$. Hence, from (3.10–11), e_x^* is the splitting edge from the vertex

$$(\mathcal{P}^*_{+x}x)_{\mathcal{P}^*} = (\mathcal{P}^*_{+x}x, \cdots, (\mathcal{P}^*_{+x})^{-1}x, \mathcal{P}^*_{+x}\gamma x, \cdots, (\mathcal{P}^*_{+x})^{-1}\gamma x)$$

on M^*.

(2) Otherwise, *i.e.*, e_x is a double edge on $M + e_x$. Thus, $\gamma x \in (x)_{\mathcal{P}_{+x}\gamma} \cup (\beta x)_{\mathcal{P}_{+x}\gamma}$. Two cases should be considered.

(i) If $\gamma x \in (x)_{\mathcal{P}_{+x}\gamma}$, then e_x is a segmentation edge on $M + e_x$. From Corollary 3.1, $\mathcal{P}^*_{+x}x$ and $\mathcal{P}^*_{+x}\gamma x$ are not transitive on M. Furthermore, from (3.12), e_x^* is a splitting edge(a harmonic loop) on M^*.

(ii) If $\gamma x \in (\beta x)_{\mathcal{P}_{+x}\gamma}$, then from (3.13), e_x^* is a splitting edge on M^*.

Sufficiency. (1) If e_x^* is not a loop on $M^* \circ e_x^*$, then from (3.10–11), M has a face

$$(\mathcal{P}^*_{\circ x}x)_{\mathcal{P}\gamma} = (\mathcal{P}^*_{\circ x}x, \cdots, (\mathcal{P}^*_{\circ x})^{-1}x, \mathcal{P}^*_{\circ x}\gamma x, \cdots, (\mathcal{P}^*_{\circ x})^{-1}\gamma x).$$

From (3.6), e_x is an appending edge between

$$\text{angle } \langle \alpha {\mathcal{P}^*_{\circ x}}^{-1}x, \mathcal{P}^*_{\circ x}\gamma x\rangle \text{ and angle } \langle \alpha {\mathcal{P}^*_{\circ x}}^{-1}\gamma x, \mathcal{P}^*_{\circ x}x\rangle$$

in a face of M, as shown in (3.8–9).

(2) Otherwise, *i.e.*, e_x^* is a loop. From (3.12–13), there are two faces on M with an angle each. Such two angles determine the appending edge e_x on M. □

Theorem 3.9 The dual of a premap $M + e_x$ is the premap $M^* \circ e_x^*$ where M^* is the dual of M and e_x^* is the dual edge in $M^* \circ e_x^*$ corresponding to e_x in $M + e_x$.

Proof A directed result of Lemma 3.4. $\square$

From what has been discussed above, both the following diagrams

$$\begin{tikzcd}
(\mathcal{X}_{\alpha,\beta}, \mathcal{P}) \arrow[r, "-e_x"] \arrow[d, leftrightarrow, "*"'] & (\mathcal{X}_{\alpha,\beta} - Kx, \mathcal{P}_{-e_x}) \arrow[d, leftrightarrow, "*"'] \\
(\mathcal{X}^*, \mathcal{P}^*) & (\mathcal{X}^* - K^*x, \mathcal{P}^*_{\bullet e_x^*}) \arrow[l, "\circ e_x^*"']
\end{tikzcd} \tag{3.14}$$

and

$$\begin{tikzcd}
(\mathcal{X}_{\alpha,\beta}, \mathcal{P}) \arrow[d, leftrightarrow, "*"'] & (\mathcal{X}_{\alpha,\beta} - Kx, \mathcal{P}_{-e_x}) \arrow[l, "+e_x"] \arrow[d, leftrightarrow, "*"'] \\
(\mathcal{X}^*, \mathcal{P}^*) \arrow[r, "\bullet e_x^*"'] & (\mathcal{X}^* - K^*x, \mathcal{P}^*_{\bullet e_x^*})
\end{tikzcd} \tag{3.15}$$

are commutative.

Example 3.3 Map $M = (Kx+Ky, \mathcal{P})$, $\mathcal{P} = (x, y)(\gamma y, \gamma x)$ and its dual $M^* = (K^*x + K^*y, \mathcal{P}^*)$,

$$\mathcal{P}^* = \mathcal{P}\gamma = (x, y, \alpha x, \gamma y)(y, \gamma x),$$

are, respectively, shown in (a) and (b) of Fig.3.10. Notice that α and β have distinguished roles in $K = \{1, \alpha, \beta, \gamma\}$ and $K^* = \{1, \beta, \alpha, \gamma\}$.

Map $M + e_z = (Kx + Ky + Kz, \mathcal{P}_{+z})$,

$$\mathcal{P}_{+z} = (x, y, z)(\gamma x, \beta z, \gamma y)$$

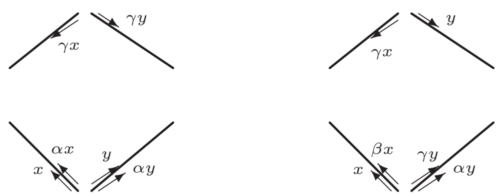
and its dual $(M + e_z)^* = (K^*x + K^*y + K^*z, (\mathcal{P}_{+z})^*)$,

$$(\mathcal{P}_{+z})^* = \mathcal{P}_{+z}\gamma = (x, \beta z, \gamma y, \gamma z)(\gamma x, \beta y),$$

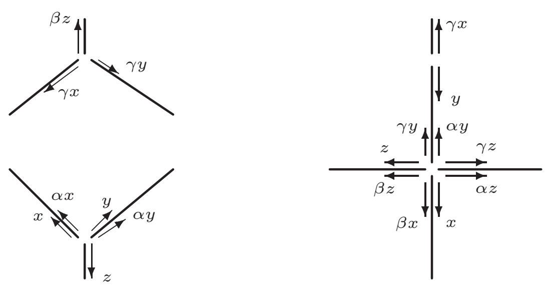
are, respectively, shown in (c) and (d) of Fig.3.10. According to (3.13), e_z^* is the splitting edge at the pair of angles $\langle x, \alpha y\rangle$ and $\langle \gamma y, \beta x\rangle$ in M^*. Therefore,

$$M^* \circ e_z^* = (K^*x + K^*y + K^*z, (z, \beta x, \alpha z, \gamma y)(y, \gamma x)).$$

If βy is seen as y, then $M^* \circ e_z^* = (M + e_z)^*$.



(a) M (b) M^*



(c) $M + e_z$ (d) $(M + e_z)^*$

Fig.3.10 Duality between appending and splitting an edge

Based on Theorem 3.9, in a premap $M = (\mathcal{X}, \mathcal{P})$, splitting an edge e_x attains $M \circ e_x$ which is just $M^* + e_x^*$ obtained by appending the edge e_x^* in its dual $M^* = (\mathcal{X}^*, \mathcal{P}^*)$.

III.4 Basic transformation

In a premap $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, the deletion of a single edge e_x is called *basic deleting* an edge and its result is denoted by $M -_{\rm b} e_x$. The contraction of a link e_x is called *basic contracting* an edge, and its result is denoted by $M \bullet_{\rm b} e_x$. The two operations are, in all, called *basic subtracting* an edge. Similarly, appending a single edge is called *basic appending* an edge, and splitting a link is called *basic splitting* an edge. Such two operations are, in all, called *basic adding* an edge. Apparently, $M +_{\rm b} e_x$ and $M \circ_{\rm b} e_x$ are the results of basic adding an edge e_x on M in their own right. Basic subtracting and basic adding an edge are in all called *basic transformation.* From what we have known above, A premap becomes another premap under basic transformation.

Theorem 3.10 Suppose M' is a premap obtained by basic transformation from premap M, then M' is a map if, and only if, M is a map.

Proof Because a single edge is never a harmonic link, from Theorem 3.4 the theorem holds for basic deleting an edge. Because a link is never a harmonic loop, from Theorem 3.5, the theorem holds for basic contracting an edge. Then, from Theorem 3.7–8, the theorem holds for basic adding an edge. $\square$

Furthermore, for basic transformation, the following conclusion can also be done.

Theorem 3.11 Let $M = (\mathcal{X}, \mathcal{P})$ be a map and $M^* = (\mathcal{X}^*, \mathcal{P}^*)$, its dual. Then, for any single edge e_x in M, $(M -_{\rm b} e_x)^* = M^* \bullet_{\rm b} e_x^*$ and for a single edge e_x not in M, $(M +_{\rm b} e_x)^* = M^* \circ_{\rm b} e_x^*$. Conversely, for any link e_x in M, $M \bullet_{\rm b} e_x = M^* -_{\rm b} e_x^*$ and for a link e_x not in M, $M \circ_{\rm b} e_x = M^* +_{\rm b} e_x^*$.

Proof Based on the duality between edges as shown in Table 3.1, the statements are meaningful. From Theorem 3.6 and Theorem 3.9, the fist statement is true. In virtue of the duality, the second

statement is true. □

From this theorem, the following two diagrams are seen to be commutative:

$$\begin{tikzcd}
(\mathcal{X}_{\alpha,\beta}, \mathcal{P}) \arrow[r, "-_{\mathrm{b}}e_x"] \arrow[d, leftrightarrow, "*"'] & (\mathcal{X}_{\alpha,\beta} - Kx, \mathcal{P}_{-_{\mathrm{b}}e_x}) \arrow[d, leftrightarrow, "*"'] \\
(\mathcal{X}^*, \mathcal{P}^*) & (\mathcal{X}^* - K^*x, \mathcal{P}^*_{\bullet_{\mathrm{b}}e_x^*}) \arrow[l, "\circ_{\mathrm{b}}e_x^*"']
\end{tikzcd} \tag{3.16}$$

and

$$\begin{tikzcd}
(\mathcal{X}_{\alpha,\beta}, \mathcal{P}) \arrow[d, leftrightarrow, "*"'] & (\mathcal{X}_{\alpha,\beta} - Kx, \mathcal{P}_{-_{\mathrm{b}}e_x}) \arrow[l, "+_{\mathrm{b}}e_x"] \arrow[d, leftrightarrow, "*"'] \\
(\mathcal{X}^*, \mathcal{P}^*) \arrow[r, "\bullet_{\mathrm{b}}e_x^*"'] & (\mathcal{X}^* - K^*x, \mathcal{P}^*_{\bullet_{\mathrm{b}}e_x^*})
\end{tikzcd} \tag{3.17}$$

On the basis of basic transformation, an equivalence can be established for classifying maps in agreement with the classification of surfaces.

Activities on Chapter III

III.5 Observations

O3.1 Observe the condition for two permutations Per_1 and Per_2 satisfying (3.1) on the same ground set.

O3.2 Given a tree(*e.g.*, the star of 5 edges), observe how many maps are there for their dual maps all having the tree as under graph.

O3.3 Write the dual of map $M = (Kx, (x, \beta x))$. If a map has its dual with the same under graph, then it is said to be *self-dual* for the graph. Observe if M is self-dual for its under graph.

O3.4 Provide a map which is cuttable and its under graph with a cut-vertex.

O3.5 How to distinguish the cuttability of a map and the separability of its under graph?

O3.6 Is the under graph of the dual of a Eulerian map bipartite? If yes, explain the why; otherwise, by an example.

O3.7 Is the dual of a preproper map(no double edge) always a preproper map? If yes, explain the why; otherwise, by an example.

O3.8 Is the dual of a proper map(each face formed by a circuit in its under graph) always a proper map? If yes, explain the why; otherwise, by an example.

O3.9 Is the dual of a polygonal map(two face with at most one edge in common) always a polygonal map? If yes, explain the why;

otherwise, by an example.

O3.10 Is the under graph of the dual of a map with 3-connected under graph still 3-connected? If yes, explain the why; otherwise, by an example.

III.6 Exercises

E3.1 Prove that the under graph of a map $M = (\mathcal{X}, \mathcal{P})$ is a tree if, and only if, its dual M^* has the following three properties:

(i) M^* has only one vertex;

(ii) For any $x \in \mathcal{X}$, x and γx are in the same orbit of $\mathcal{P}^*$;

(iii) For any $y \in (x)_{\mathcal{P}^*}$, there is no subsequence $x, y, \gamma x, \gamma y$ or $x, \gamma y, \gamma x, y$ in its cycle.

For a map $M = (\mathcal{X}(X), \mathcal{P})$ and $S \subseteq X$, let $M[K_S] = (K_S, \mathcal{P}[K_S])$ where $\mathcal{P}[K_S]$ is the restriction of $\mathcal{P}$ on $K_S = KS$, and $M[K_S]$ is said to be *induced* on K_S from M. Generally speaking, $M[K_S]$ is not a map, but always a premap. A cocircuit of a graph with all of its edges incident to the same vertex is called a *proper cocircuit* .

E3.2 Let $M - E_S = M[\mathcal{X} - K_S]$, $E_S = \{e_x | \forall x \in S\}$. Prove that $M - E_S$, $S \subseteq X$, is a map if, and only if, there is no proper cocircuit of $G[M]$ on graph $G(M[K_S])$.

A *proper circuit* of a map M is such a set C of edges that $C^* = \{e^* | \forall e \in C\}$ is a proper cocircuit of $G[M^*]$.

E3.3 Let $H = G[M]$ and $\mathcal{M}_H$ be the set of all maps whose under graphs are H. Prove that if C is a proper circuit of a map M, then it is a proper circuit of all $N \in \mathcal{M}_H$.

E3.4 Let $M \bullet E_S = (M^* - E^*_S)^*$ where M^* is the dual of M and $E^*_S = \{e^*_x | \forall x \in S\}$. Prove that $M \bullet E_S$ is a map if, and only if, E_S is a proper circuit of M.

E3.5 Prove that a map M is on the sphere if, and only if, each face of its dual M^* corresponds to a proper cocircuit of M.

If a map has its dual Eulerian, then it is called a *dual Eulerian map*.

E3.6 Prove that a map is a dual Eulerian map if, and only if, each of its faces is incident with even number of edges.

If a preproper dual Eulerian map has each of its faces partitionable into circuits, then it is called a *even assigned map* . A map is called *bipartite* when its under graph is bipartite.

E3.7 Prove that a dual Eulerian map is bipartite if, and only if, it is even assigned.

E3.8 Prove that a quadrangulation is bipartite if, and only if, it is without loop.

For a loopless quadrangulation $Q = (\mathcal{X}_{\alpha,\beta}, \mathcal{Q})$, from E3.8, its vertex set V can be partitioned into two subsets V_1 and V_2 such that the two ends of each edge are never in the same subset. Such a subset of vertices is called an *independent set*. $\mathcal{X}^{(1)}_{\alpha_1,\beta_1}$ and $\mathcal{X}^{(2)}_{\alpha_2,\beta_2}$ stand for the sets of elements in $\mathcal{X}_{\alpha,\beta}$ incident to, respectively, V_1 and V_2[Deh1, Gau1, MuS1].

E3.9 Let $\sigma = \beta\mathcal{Q}\gamma$, $\gamma = \alpha\beta$. Prove that $(\sigma x, \sigma\mathcal{Q}\alpha x)$, $x \in \mathcal{X}_{\alpha,\beta}$, is an angle.

Angles $(\sigma x, \sigma\mathcal{Q}\alpha x)$ and $(x, \mathcal{Q}\alpha x)$ as shown in E3.9 are called an *independent pair* . Thus, each face in a quadrangulation has exactly two independent pairs of angles.

E3.10 Let $K_1 = \{1, \alpha_1, \beta_1, \gamma_1\}$, $\gamma_1 = \alpha_1\beta_1$. And, $\alpha_1 = \mathcal{Q}$ and $\beta_1 = \beta\mathcal{Q}\gamma (= \sigma$ as shown in E3.9). Prove

(i) K_1 is the Klein group of four elements;

(ii) $\mathcal{X}^{(1)}_{\alpha_1,\beta_1} = \sum_{v_x \in V_1} \sum_{y \in \{x\}_{\mathcal{Q}}} K_1 y$;

(iii) $Q_1 = (\mathcal{X}^{(1)}_{\alpha_1,\beta_1}, \mathcal{Q}_1^{-1})$ is a map, where $\mathcal{Q}_1$ is the restriction of $\mathcal{Q}$ on $\mathcal{X}^{(1)}_{\alpha_1,\beta_1}$.

Similarly to E3.10, from V_2, another map Q_2 can be deduced. Q_1 and Q_2 are called the *incident pair* of the quadrangulation.

E3.11 Prove that the two maps in the incident pair of a quadrangulation are mutually dual.

E3.12 Prove that any planar quadrangulation is loopless.

E3.13 Let $\mathcal{A}$ and $\mathcal{B}$ be the sets of, respective all planar quadrangulations and all dual pairs of planar maps. Establish a 1–to–1 correspondence between $\mathcal{A}$ and $\mathcal{B}$(*i.e.*, *bijection*).

III.7 Researches

For a map, if the basic deletion of an edge can not be done anymore, then the map is said to be *basic deleting edge irreducible*. Similarly, if the basic contraction of an edge can not be done on a map anymore, then the map is said to be *basic contracting irreducible*.

R3.1 Given the size, determine the number of self-dual maps as an integral function of the size, or provide a way to list all the self-dual maps of the same size and deduce a relation among the numbers of different sizes.

R3.2 Given the size, determine the number of maps all basic deleting irreducible as an integral function of the size, or provide a way to list all such maps with the same size and deduce a relation among the numbers of different sizes.

R3.3 Given the size, determine the number of maps all basic contracting irreducible as an integral function of the size, or provide a way to list all such maps with the same size and deduce a relation among the numbers of different sizes.

R3.4 For any given graph, determine the number of maps all basic deleting irreducible with the same under graph, or provide a way to list all such maps with the same size and deduce a relation among the numbers of different sizes.

R3.5 For any given map, determine the number of all basic deleting irreducible maps obtained from the map by basic deletion, or

provide a way to list all such maps with the same size and deduce a relation among the numbers of different sizes.

R3.6 For a given graph, determine the number of maps all basic contracting irreducible with the same under graph, or provide a way to list such maps and deduce a relation among the numbers of different sizes.

R3.7 For a given map, determine the number of all basic contracting irreducible maps obtained from the map by basic contraction, or provide a way to list such maps and deduce a relation among the numbers of different sizes.

If a map is basic both deleting and contracting irreducible, then it is said to be *basic subtracting irreducible*.

R3.8 Given the size, determine the number of basic subtracting irreducible maps as an integral function of the size, or provide a way to list all such maps with the same size and deduce a relation among the numbers of different sizes.

R3.9 For a given graph, determine the number of maps all basic subtracting irreducible with the same under graph, or provide a way to list such maps and deduce a relation among the numbers of different sizes.

R3.10 Find a relation between triangulations and quadrangulations.

If a map has each of its faces pentagon, then it is called a *quinquangulation*. Similarly, the meaning of a *hexagonalization*.

R3.11 Justify whether or not a triangulation has a spanning quinquangulation or hexagonalization. If do, determine its number.

R3.12 *Even assigned conjecture*: A bipartite graph without cut-edge has a super map even assigned.

Chapter IV

Orientability

- The orientability is determined by the orientation for each edge with two sides; otherwise, nonorientability.
- The basic equivalence is defined via basic transformations to show that the orientability is an invariant in an equivalent class. This equivalence is, in fact, the elementary equivalence on surfaces.
- The Euler characteristic is also shown to be an invariant in an equivalent class.
- Two examples show that none of the orientability(nonorientability as well) and the Euler characteristic can determine the equivalent class.

IV.1 Orientation

Let $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ be a map(from Theorem 2.6, without loss of generality for a premap), and Ψ_I, $I = \{\gamma, \mathcal{P}\}$, $\gamma = \alpha\beta = \beta\alpha$, be the group generated by the set of permutations I. Now, it is known that the number of orbits of $\mathcal{P}$ on $\mathcal{X}_{\alpha,\beta}$ is double the number of vertices on M and the number of orbits of $\mathcal{P}\gamma$ on $\mathcal{X}_{\alpha,\beta}$ is double the number of faces on M. Because $\mathcal{P}, \mathcal{P}\gamma \in \Psi_I$, the number of orbits of the group Ψ_I on $\mathcal{X}_{\alpha,\beta}$ is not greater than any of their both.

Lemma 4.1 The number of orbits of the group Ψ_I on $\mathcal{X}_{\alpha,\beta}$ is not greater than 2.

Proof Because $\mathcal{P}\gamma \in \Psi_I$, for any $x \in \mathcal{X}_{\alpha,\beta}$, $\{x\}_{\mathcal{P}\gamma} \subseteq \{x\}_{\Psi_I}$. Here, $\{x\}_{\mathcal{P}\gamma}$ and $\{x\}_{\Psi_I}$ are the orbits of, respectively, the permutation $\mathcal{P}\gamma$ and the group Ψ_I on $\mathcal{X}_{\alpha,\beta}$. For any chosen element $x \in \mathcal{X}_{\alpha,\beta}$, from $\mathcal{P} \in \Psi$, for any $y \in \{x\}_{\mathcal{P}\gamma}$, $\{y\}_{\mathcal{P}\gamma} \subseteq \{x\}_{\Psi_I}$, and from $\gamma \in \Psi$,

$$\{\gamma y\}_{\mathcal{P}\gamma} \subseteq \{\gamma y\}_{\Psi_I} \subseteq \{x\}_{\Psi_I}.$$

In view of Theorem 2.6, at least half of elements at each vertex belong to $\{x\}_{\Psi_I}$. Therefore, $\{x\}_{\Psi_I}$ contains at least half of elements in $\mathcal{X}_{\alpha,\beta}$.

Similarly, $\{\alpha x\}_{\Psi_I}$ contains at least half of elements in $\mathcal{X}_{\alpha,\beta}$.

In consequence, based on the basicness of $\mathcal{P}$ for α, Ψ_I has at most 2 orbits on $\mathcal{X}_{\alpha,\beta}$. $\square$

According to this lemma, a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ has only two possibilities: group Ψ_I is with two, or one, orbits on $\mathcal{X}_{\alpha,\beta}$. The former is called *orientable*, and the later, *nonorientable* .

From the proof of the lemma, an efficient algorithm can be established for determining all the orbits of group Ψ_I on the ground set.

Actually, in an orientable map, because Ψ_I has two orbits for α, the ground set is partitioned into two parts of equal size. It is seen from Lemma 4.1 that each quadricell (*i.e.*, edge) is distinguished by two elements in each of the two orbits. And, the two elements of an edge in the same orbit have to be with different ends of the edge. Thus, each of the two orbits determines the under graph of the map.

Example 4.1 Consider map $M = (\mathcal{X}, \mathcal{P})$ where

$$\mathcal{X} = Kx + Ky + Kz + Ku + Kv + Kw$$

and

$$\mathcal{P} = (x, y, z)(\gamma z, u, v)(\gamma v, \gamma y, w)(\gamma w, \gamma u, \gamma x)$$

as shown in Fig.4.1(a). Its two faces are

$$(x, \gamma w, \gamma v, \gamma z)$$

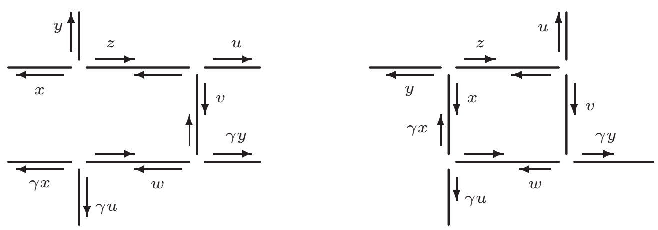
and

$$(\gamma x, y, w, \gamma u, v, \gamma y, z, u).$$

In fact, for this map, group Ψ_I has two orbits. One is

$$\{x, \gamma w, \gamma v, \gamma z, \gamma x, y, w, \gamma u, v, \gamma y, z, u\}.$$

The other is what is obtained from it by multiplying α to each of all its elements. Thus, M is orientable. Fig.4.1(b) shows that M is an embedding of the complete graph of order 4 on the torus $(yuy^{-1}u^{-1})$.



(a) (b)

Fig.4.1 An embedding of K_4

Corollary 4.1 If Ψ_I, $I = \{\mathcal{P}, \alpha\beta\}$, has two orbits on $\mathcal{X}_{\alpha,\beta}$, then they are conjugate for both α and β.

Proof It is known from Lemma 4.1 that the two obits have the same number of elements, *i.e.*, half of $\mathcal{X}_{\alpha,\beta}$. Because $y \in \{x\}_{\Psi_I}$ if, and only if, $\alpha y \in \{\alpha x\}_{\Psi_I}$ and for any Kx, αx and βx are always in the same orbit of Ψ_I, this implies that $\{\alpha x\}_{\Psi_I} = \{\beta x\}_{\Psi_I}$ different from $\{x\}_{\Psi_I}$ and hence the conclusion. □

Example 4.2 Consider map $N = (\mathcal{X}, \mathcal{Q})$ where

$$\mathcal{X} = Kx + Ky + Kz + Ku + Kv + Kw$$

and

$$\mathcal{Q} = (x, y, z)(\gamma z, u, v)(\gamma v, \beta y, w)(\gamma w, \gamma u, \gamma x)$$

as shown in Fig.4.2(a).

That is obtained from the map M in Fig.4.1(a) in the replacement of cycle $(\gamma v, \gamma y, w)$ by cycle $(\gamma v, \beta y, w)$. Here, N has also two faces

$$(x, \gamma w, \gamma v, \gamma z)$$

and

$$(\gamma x, y, \beta v, \alpha u, \beta w, \gamma y, z, u).$$

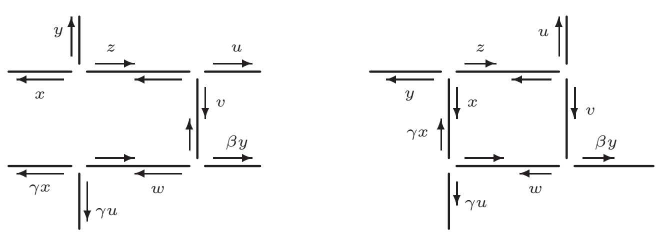
Because $\beta y \in \{y\}_{\mathcal{Q}\gamma} \subseteq \{y\}_{\Psi_{\{\gamma,\mathcal{Q}\}}}$, from the corollary group $\Psi_{\{\gamma,\mathcal{Q}\}}$ has only one orbit, *i.e.*,

$$\{x\}_{\Psi_{\{\gamma,\mathcal{Q}\}}} = \mathcal{X}.$$

Therefore, N is nonorientable. It is seen from Fig.4.2(b) that N is an embedding of the complete graph of order 4 on the surface

$$(yuyu^{-1}) \sim_{\text{top}} (yyuu),$$

i.e., Klein bottle .



(a) (b)

Fig.4.2 An embedding of K_4 on the Klein bottle

Theorem 4.1 A map $M = (\mathcal{X}, \mathcal{P})$ is nonorientable if, and only if, there exists an element $x \in \mathcal{X}$ such that $\beta x \in \{x\}_{\Psi_I}$, or $\alpha x \in \{x\}_{\Psi_I}$ where $I = \{\gamma, \mathcal{P}\}$.

Proof Necessity. Suppose $\alpha x \notin \{x\}_{\Psi_I}$, then Ψ_I has at least two orbits. However from Lemma 4.1, it has exactly two orbits . Thus, M is never nonorientable. The necessity holds

Sufficiency. Because $\beta x \in \{x\}_{\Psi_I}$, from Corollary 4.1 it is only possible to have $\{x\} = \mathcal{X}$, *i.e.*, Ψ_I has only one orbit. Hence, M is nonorientable. This is the sufficiency. $\square$

This theorem enables us to justify the nonorientability and hence the orientability of a map much simple. If there exists a face $(x)_{\mathcal{P}\gamma}$, denoted by $\mathcal{S}_x$, such that $\alpha x \in \mathcal{S}_x$, or there exists a vertex $(x)_{\mathcal{P}}$, denoted by $\mathcal{S}_x$, such that $\beta x \in \mathcal{S}_x$ on M, then M is nonorientable(as shown in Example 2). Otherwise, From $y \in \mathcal{S}_x$ via acting $\mathcal{P}$, or γ, for getting $z \notin \mathcal{S}_x$, $\mathcal{S}_x$ is extended into

$$\mathcal{S}_x \cup \{z\}_{\mathcal{P}\gamma} \cup \{z\}_{\mathcal{P}}$$

which is seen as a new $\mathcal{S}$ to see if $y, \alpha y \in \mathcal{S}$, or $y, \beta y \in \mathcal{S}$. If it does, then M is nonorientable; otherwise, do the extension until $|\mathcal{S}| = |\mathcal{X}|/2$, or $\mathcal{S} = \mathcal{X}$.

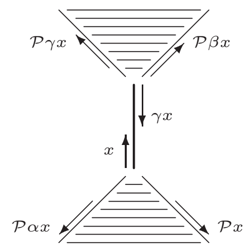
Theorem 4.2 A map $M = (\mathcal{X}, \mathcal{P})$ is orientable if, and only if, its dual $M^* = (\mathcal{X}^*, \mathcal{P}^*)$ is orientable.

Proof Because $\mathcal{P}^* = \mathcal{P}\gamma \in \Psi_{\{\gamma,\mathcal{P}\}}(\gamma = \alpha\beta = \beta\alpha)$, $\Psi_{\{\gamma,\mathcal{P}\}} = \Psi_{\{\gamma,\mathcal{P}^*\}}$. So, for any $x \in \mathcal{X} = \mathcal{X}^*$, $\{x\}_{\Psi_{\{\gamma,\mathcal{P}\}}} = \{x\}_{\Psi_{\{\gamma,\mathcal{P}^*\}}}$. This is the conclusion of the theorem. $\square$

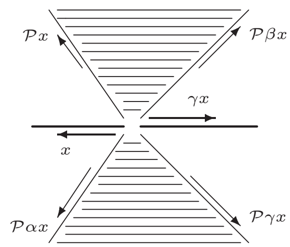
IV.2 Basic equivalence

First, observe the effect for the orientability of a map via basic transformation.

For a map $M = (\mathcal{X}, \mathcal{P})$ and its edge e_x, let $M -_{\mathrm{b}} e_x$ and $M \bullet_{\mathrm{b}} e_x$ be, respectively, obtained by basic deleting and basic contracting the edge e_x on M. From Theorem 3.10, $M -_{\mathrm{b}} e_x$ and $M \bullet_{\mathrm{b}} e_x$ are both a map .



(a) Single link e_x



(b) Single loop e_x

Fig.4.3 Basic deleting an edge

Lemma 4.2 If M' is the map obtained by basic subtracting an edge from M, then M' is orientable if, and only if, M is orientable.

Proof First, to prove the theorem for $M' = M -_{\mathrm{b}} e_x$.

Necessity. From $M' = M - e_x = (\mathcal{X}', \mathcal{P}')$ orientable, group $\Psi' = \Psi_{\{\gamma,\mathcal{P}'\}}$ has two orbits on $\mathcal{X}' = \mathcal{X} - Kx$, *i.e.*, $\{\mathcal{P}x\}_{\Psi'}$ and $\{\mathcal{P}\alpha x\}_{\Psi'}$. Because e_x is single, $\mathcal{P}\gamma x \in \{\mathcal{P}x\}_{\Psi'}$ and $\mathcal{P}\beta x \in \{\mathcal{P}\alpha x\}_{\Psi'}$. So, group $\Psi = \Psi_{\{\gamma,\mathcal{P}\}}$ has two orbits

$$\{x\}_{\Psi} = \{\mathcal{P}x\}_{\Psi'} \cup \{x, \gamma x\}$$

and

$$\{\alpha x\}_{\Psi} = \{\mathcal{P}\alpha x\}_{\Psi'} \cup \{\alpha x, \beta x\}$$

on $\mathcal{X}$, *i.e.*, M is orientable.

Sufficiency. Because e_x is a single link (Fig.4.3(a)), or single loop (Fig.4.3(b)), in virtue of that group Ψ has two orbits $\{x\}_{\Psi}$ and $\{\alpha x\}_{\Psi}$ on $\mathcal{X}$, group Ψ' has two orbits

$$\{\mathcal{P}x\}_{\Psi'} = \{x\}_{\Psi} - \{x, \gamma x\}$$

and

$$\{\mathcal{P}\alpha x\}_{\Psi'} = \{\alpha x\}_{\Psi} - \{\alpha x, \beta x\}$$

on $\mathcal{X}'$, *i.e.*, M' is orientable.

Then, to prove the theorem for $M' = M \bullet_{\mathrm{b}} e_x$. On the basis of Theorem 3.11, the result is directly deduced from that for $M' = M -_{\mathrm{b}} e_x$. $\square$

Whenever that two new angles occur in the deletion of an edge with 4 angles lost is noticed, the edge appending as the inverse of deletion is done between the two angles. And then the same case comes for basic deleting and basic appending an edge. In this sense, Lemma 4.3 in what follows is seen as a direct result of Lemma 4.2. However, it is still proved in an independent way.

For basic appending an edge, since the edge is only permitted to be a single link or a single loop, this operation is, in fact, done by putting the edge in the same face.

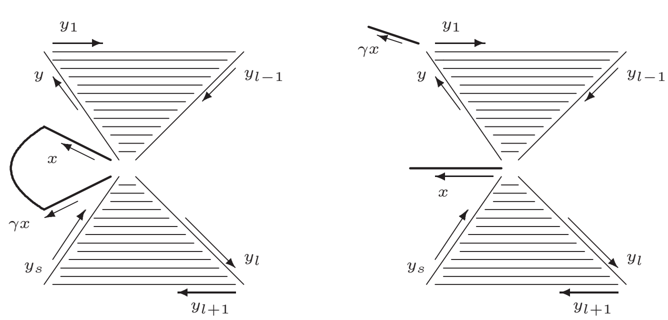
Let map $M = (\mathcal{X}, \mathcal{P})$ have a face

$$(y)_{\mathcal{P}\gamma} = (y_0, y_1, \cdots, y_s)$$

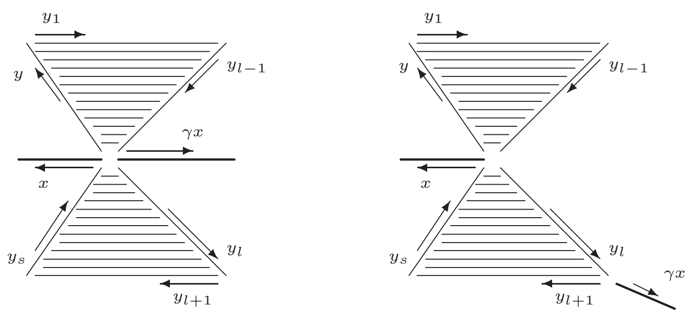
where $y_0 = y$, $y_1 = (\mathcal{P}\gamma)y$, $\cdots$, $y_s = (\mathcal{P}\gamma)^{-1}y$. Denote

$$M +_i e_x = M +_{\mathrm{b}} e_x$$

when appending the edge e_x in between angles $\langle y, \mathcal{P}\alpha y\rangle$ and $\langle y_i, \mathcal{P}\alpha y_i\rangle$, $0 \le i \le s$. From (3.10), $M +_i e_x = M +_{\mathrm{b}} e_x$, $0 \le i \le s$, are all maps(Fig.4.4).



(a) $M +_0 e_x$ (b) $M +_1 e_x$



(c) $M +_l e_x$ (d) $M +_{l+1} e_x$

Fig.4.4 Basic appending an edge

Lemma 4.3 Maps $M+_i e_x = M+_\mathrm{b} e_x$, $0 \leq i \leq s$, are orientable if, and only if, M is orientable.

Proof necessity. Since $M' = M +_i e_x = (\mathcal{X}', \mathcal{P}')$, $0 \leq i \leq s$, are all orientable, group $\Psi' = \Psi_{\{\gamma,\mathcal{P}'\}}$ has two orbits $\{x\}_{\Psi'}$ and $\{\alpha x\}_{\Psi'}$ on $\mathcal{X}' = \mathcal{X} + Kx$. Because e_x is a single link (Fig.4.4(a) and (c)), or single loop (Fig.4.4(b) and (d)),

$$\mathcal{P}'x \in \{x\}_{\Psi'} \text{ and } \mathcal{P}'\alpha x \in \{\alpha x\}_{\Psi'}.$$

Hence, group $\Psi = \Psi_{\{\gamma,\mathcal{P}\}}$ has two orbits

$$\{\mathcal{P}'x\}_\Psi = \{x\}_{\Psi'} - \{x, \gamma x\}$$

and

$$\{\mathcal{P}'\alpha x\}_\Psi = \{\alpha x\}_{\Psi'} - \{\alpha x, \beta x\}$$

on $\mathcal{X}$. This implies that M is orientable.

Sufficiency. Since e_x is a single link (Fig.4.4(a) and (c)), or single loop (Fig.4.4(b) and (d)), the two orbits

$$\{x\}_{\Psi'} = \{y\}_\Psi + \{x, \gamma x\}$$

and

$$\{\alpha x\}_{\Psi'} = \{\alpha y\}_\Psi + \{\alpha x, \beta x\}$$

of group Ψ' on $\mathcal{X}'$ are deduced from the two orbits $\{y\}_\Psi$ and $\{\alpha y\}_\Psi$ of group Ψ on $\mathcal{X}$. Therefore, M' is orientable. $\square$

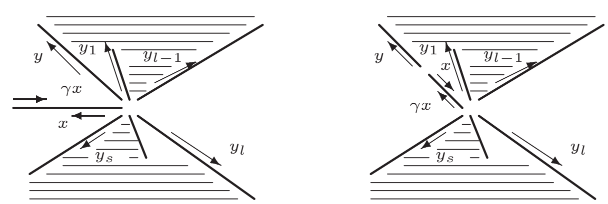
As for basic splitting an edge, whenever that two new angles occur in the contraction of an edge with 4 angles lost is noticed, the edge splitting seen as the inverse of contraction is done between the two angles.

Next, consider how to list all possibilities for basic splitting from a given angle.

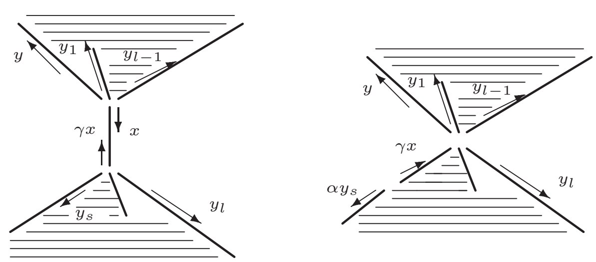
For a map $M = (\mathcal{X}, \mathcal{P})$, let

$$(y)_{\mathcal{P}} = (y_0, y_1, \cdots, y_{l-1}, y_l, \cdots, y_s),$$

$y_0 = y$, $s \geq 0$, be a vertex. Denote by $M \circ_i e_x$ the result obtained from M by basic splitting an edge between angles $\langle y, \mathcal{P}\alpha y\rangle$ and $\langle y_i, \alpha y_{i-1}\rangle$ where $y = y_0$ and $\mathcal{P}\alpha y = \alpha y_s$. From Theorem 3.10, $M \circ_i e_x = M +_{\mathrm{b}} e_x$, $0 \leq i \leq s$, are all maps (Fig.4.5).



(a) $M \circ_0 e_x$ (b) $M \circ_1 e_x$



(c) $M \circ_l e_x$ (d) $M \circ_s e_x$

Fig.4.5 Basic splitting an edge

Lemma 4.4 For a map $M = (\mathcal{X}, \mathcal{P})$ and $x \notin \mathcal{X}$, map $M \circ_i e_x = M +_{\mathrm{b}} e_x$, $0 \leq i \leq s$, is orientable if, and only if, M is orientable.

Proof Necessity. Because $M' = M \circ_i e_x = (\mathcal{X}', \mathcal{P}')$, $0 \leq i \leq s$, is orientable, group $\Psi' = \Psi_{\{\gamma, \mathcal{P}'\}}$ has two obits $\{x\}_{\Psi'}$ and $\{\alpha x\}_{\Psi'}$ on

$\mathcal{X}' = \mathcal{X} + Kx$. Since e_x is a single link (Fig.4.5(b), (c) and (d)), or a double link (Fig.4.5(a) and (c)),

$$\mathcal{P}'x \in \{x\}_{\Psi'} \text{ and } \mathcal{P}'\alpha x \in \{\alpha x\}_{\Psi'}.$$

Therefore, group $\Psi = \Psi_{\{\gamma,\mathcal{P}\}}$ has two orbits

$$\{\mathcal{P}'x\}_{\Psi} = \{x\}_{\Psi'} - \{x, \gamma x\} \text{ and } \{\mathcal{P}'\alpha x\}_{\Psi} = \{\alpha x\}_{\Psi'} - \{\alpha x, \beta x\}$$

on $\mathcal{X}$. This implies that M is orientable.

Sufficiency. Because e_x is a single link (Fig.4.5(b), (c) and (d)), or a double link (Fig.4.5(a) and (c)), the two orbits $\{x\}_{\Psi'} = \{y\}_{\Psi} + \{x, \gamma x\}$ and $\{\alpha x\}_{\Psi'} = \{\alpha y\}_{\Psi} + \{\alpha x, \beta x\}$ of group Ψ' on $\mathcal{X}'$ are deduced from the two orbits $\{y\}_{\Psi}$ and $\{\alpha y\}_{\Psi}$ of group Ψ on $\mathcal{X}$. Therefore, M' is orientable . □

Corollary 4.2 If M' is the map obtained by basic adding an edge from map M, then M' is orientable if, and only if, M is orientable.

Proof A direct result of Lemma 4.3 and Lemma 4.4. □

The operation of basic appending an edge between two successive angles of a face in a map is also called *increasing duplition* (Fig.4.4(b) and (d)), and its inverse operation, *decreasing duplition*. And dually, the operation of basic splitting an edge is also called *increasing subdivision*(Fig.4.5(b) and (d)), and its inverse operation, *decreasing subdivision*.

Corollary 4.3 A premap M' obtained by increasing duplition, increasing subdivision, decreasing duplition, or decreasing subdivision from a map M is still a map with the same orientability of M.

Proof The results for decreasing duplition and decreasing subdivision are derived from Lemma 4.2. Those for increasing duplition and increasing subdivision are from Corollary 4.2. □

If map M_1 can be obtained from map M_2 via a series of basic adding and/or basic subtracting an edge, then they are called mutually *basic equivalence*, denoted by $M_1 \sim_{\mathrm{bc}} M_2$.

Theorem 4.3 If maps $M_1 \sim_{\rm bc} M_2$, then M_1 is orientable if, and only if, M_2 is orientable.

Proof A direct result of Lemma 4.2 and Corollary 4.2. □

Since $\sim_{\rm bc}$ is an equivalent relation, maps are partitioned into *classes of basic equivalence*, in short *equivalent class*. Theorem 4.3 shows that the orientability of maps is an invariant in the same equivalent class.

IV.3 Euler characteristic

For a map $M = (\mathcal{X}, \mathcal{P})$, let $\nu = \nu(M)$, $\epsilon = \epsilon(M)$ and $\phi = \phi(M)$ are, respectively, the order(vertex number), size(edge number) and *co-order*(face number) of M, then

$$\chi(M) = \nu - \epsilon + \phi \tag{4.1}$$

is called the *Euler characteristic* of M.

Theorem 4.4 Let M^* be the dual of a map M, then

$$\chi(M^*) = \chi(M). \tag{4.2}$$

Proof Because $\nu(M^*) = \phi(M)$, $\epsilon(M^*) = \epsilon(M)$ and $\phi(M^*) = \nu(M)$, (4.2) is obtained from (4.1). □

Lemma 4.5 For a map $M = (\mathcal{X}, \mathcal{P})$ and an edge e_x, $x \in \mathcal{X}$, let $M - e_x$ and $M \bullet e_x$ be, respectively, obtained from M by deleting and contracting the edge e_x, then

$$\chi(M) = \begin{cases} \chi(M - e_x), & \text{if } e_x \text{ is single;} \\ \chi(M \bullet e_x), & \text{if } e_x \text{ is a link.} \end{cases} \tag{4.3}$$

Proof From Theorem 3.11 and Theorem 4.4, only necessary to consider for one of $M - e_x$ and $M \bullet e_x$. Here, the former is chosen. To prove $\chi(M - e_x) = \chi(M)$ for e_x single.

Because e_x is single, $\nu(M-e_x)=\nu(M)$, $\epsilon(M-e_x)=\epsilon(M)-1$ and $\phi(M-e_x)=\phi(M)-1$. From (4.1),

$$\begin{aligned}\chi(M-e_x)&=\nu(M)-(\epsilon(M)-1)+(\phi(M)-1)\\&=\nu(M)-\epsilon(M)+\phi(M)\\&=\chi(M).\end{aligned}$$

This is just what is wanted to get. □

Corollary 4.4 For any map M, $\chi(M)\leq 2$.

Proof By induction on the co-order $\phi(M)$. If M has only one face, *i.e.*, $\phi(M)=1$, then

$$\chi(M)=\nu(M)-\epsilon(M)+1.$$

In view of the connectedness,

$$\epsilon(M)\geq\nu(M)-1.$$

In consequence,

$$\chi(M)\leq\nu(M)-(\nu(M)-1)+1=2.$$

Thus, the conclusion is true for $\phi(M)=1$.

In general, *i.e.*, $\phi(M)\geq 2$. Because of the transitivity on a map, there exists a single edge e_x on M. From Lemma 4.5, $M'=M-e_x$ has $\chi(M')=\chi(M)$. Since $\phi(M')=\phi(M)-1$, by the induction hypothesis $\chi(M')\leq 2$. That is $\chi(M)\leq 2$, the conclusion. □

For an indifferent reception, because the order, size and co-order of a map can be much greater as the map is much enlarged. The conclusion would be unimaginable. In fact, since the deletion od a single edge does not change the connectivity with the Euler characteristic unchange and the size of a connected graph is never less than its order minus one, this conclusion becomes reasonable.

Corollary 4.5 For basic subtracting an edge e_x on a map M, $\chi(M-_{\mathrm{b}}e_x)=\chi(M)$ and $\chi(M\bullet_{\mathrm{b}}e_x)=\chi(M)$.

Proof A direct result of Lemma 4.5. □

Lemma 4.6 For a map $M = (\mathcal{X}, \mathcal{P})$ and an edge e_x, $x \notin \mathcal{X}$, let $M + e_x$ and $M \circ e_x$ be obtained from M via, respectively, appending and splitting the edge e_x, then

$$\chi(M) = \begin{cases} \chi(M + e_x), \text{ if } e_x \text{ is single;} \\ \chi(M \circ e_x), \text{ if } e_x \text{ is a link.} \end{cases} \tag{4.4}$$

Proof From Theorem 3.11 and Theorem 4.4, only necessary to consider for one of $M + e_x$ and $M \circ e_x$. The former is chosen. To prove $\chi(M + e_x) = \chi(M)$.

Because e_x is single, then $\nu(M + e_x) = \nu(M)$, $\epsilon(M + e_x) = \epsilon(M) + 1$ and $\phi(M + e_x) = \phi(M) + 1$. From (4.1),

$$\begin{aligned} \chi(M + e_x) &= \nu(M) - (\epsilon(M) + 1) + (\phi(M) + 1) \\ &= \nu(M) - \epsilon(M) + \phi(M) \\ &= \chi(M). \end{aligned}$$

Therefore, the lemma is true. □

Corollary 4.6 For basic adding an edge e_x on a map M, $\chi(M +_{\mathrm{b}} e_x) = \chi(M)$ and $\chi(M \circ_{\mathrm{b}} e_x) = \chi(M)$.

Proof A direct result of Lemma 4.6. □

For a map $M = (\mathcal{X}, \mathcal{P})$ and an edge e_x, $x \in \mathcal{X}$, let $M_{[\circ x]}$ and $M_{[+x]}$ be obtained from M by, respectively, increasing subdivision and increasing duplition for edge e_x, and $M_{[\bullet x]}$ and $M_{[-x]}$, by, respectively, decreasing subdivision and decreasing duplition for edge e_x. From Corollary 4.3, they are all maps.

Corollary 4.7 For increasing subdivision and increasing duplition,

$$\chi(M_{[\circ x]}) = \chi(M); \ \chi(M_{[+x]}) = \chi(M) \tag{4.5}$$

and for decreasing subdivision and decreasing duplition,

$$\chi(M_{[\bullet x]}) = \chi(M); \ \chi(M_{[-x]}) = \chi(M). \tag{4.6}$$

Proof Because increasing subdivision and increasing duplition are a special type of basic adding an edge, from Corollary 4.5, (4.5) holds. Because decreasing subdivision and decreasing duplition are a special type of basic subtracting an edge, from Corollary 4.6, (4.6) holds. The corollary is obtained. □

The following theorem shows that the Euler characteristic is an invariant in the basic equivalent classes of maps.

Theorem 4.5 If maps $M_1 \sim_{\mathrm{bc}} M_2$, then

$$\chi(M_1) = \chi(M_2). \tag{4.7}$$

Proof Because the basic transformation consists of basic subtracting and basic adding an edge, from Corollary 4,5 and Corollary 4.6, (4.7) is obtained. □

IV.4 Pattern examples

Pattern 4.1 Consider the map $M = (\mathcal{X}, \mathcal{P})$ where $\mathcal{X} = Kx + Ky + Kz + Ku + Kw + Kl$ and

$$\mathcal{P} = (x, y, z)(\alpha l, \gamma z, \beta w)(\beta l, \gamma y, \alpha u)(w, \gamma u, \beta x),$$

shown in Fig.1.13.

By deleting the single edge e_x on M, let $M_1 = (\mathcal{X}_1, \mathcal{P}_1) = M -_{\mathrm{b}} e_x$, then $\mathcal{X}_1 = Ky + Kz + Ku + Kw + Kl$ and

$$\mathcal{P}_1 = (y, z)(\alpha l, \gamma z, \beta w)(\beta l, \gamma y, \alpha u)(w, \gamma u).$$

By contracting the double link e_z on M_1, let $M_2 = (\mathcal{X}_2, \mathcal{P}_2) = M_1 \bullet_{\mathrm{b}} e_z$, then $\mathcal{X}_2 = Ky + Ku + Kw + Kl$ and

$$\mathcal{P}_2 = (y, \beta w, \alpha l)(\beta l, \gamma y, \alpha u)(w, \gamma u).$$

By contracting the double link e_l on M_2, let $M_3 = (\mathcal{X}_3, \mathcal{P}_3) = M_2 \bullet_{\mathrm{b}} e_l$, then $\mathcal{X}_3 = Ky + Ku + Kw$ and $\mathcal{P}_3 = (y, \beta w, \gamma y, \alpha u)(w, \gamma u)$.

By contracting the double link e_u on M_3, let $M_4 = (\mathcal{X}_4, \mathcal{P}_4) = M_3 \bullet_{\mathrm{b}} e_u$, then $\mathcal{X}_4 = Ky + Kw$ and $\mathcal{P}_4 = (y, \beta w, \gamma y, \alpha w)$.

Now, M_4 has only one vertex and only one face and hence any basic transformation for subtracting an edge can not be done. It is a map on the torus (Fig.4.6).

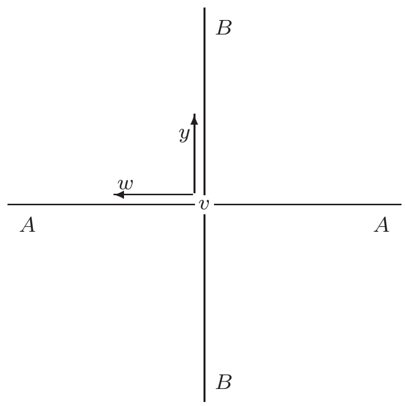


Fig.4.6 A map basic equivalent to M

Pattern 4.2 Again, consider the map $N = (\mathcal{X}, \mathcal{Q})$ where $\mathcal{X} = Kx + Ky + Kz + Ku + Kw + Kl$ and

$$\mathcal{Q} = (x, y, z)(\alpha l, \gamma z, \beta w)(\beta l, \beta y, \alpha u)(w, \gamma u, \beta x),$$

as shown in Fig.1.14.

By deleting the single edge e_x on M, let $N_1 = (\mathcal{X}_1, \mathcal{Q}_1) = N -_{\mathrm{b}} e_x$, then $\mathcal{X}_1 = Ky + Kz + Ku + Kw + Kl$ and

$$\mathcal{Q}_1 = (y, z)(\alpha l, \gamma z, \beta w)(\beta l, \beta y, \alpha u)(w, \gamma u).$$

By contracting the double link e_z on N_1, let $N_2 = (\mathcal{X}_2, \mathcal{Q}_2) = N_1 \bullet_{\mathrm{b}} e_z$, then $\mathcal{X}_2 = Ky + Ku + Kw + Kl$ and

$$\mathcal{Q}_2 = (y, \beta w, \alpha l)(\beta l, \beta y, \alpha u)(w, \gamma u).$$

By contracting the double link e_l on N_2, let $N_3 = (\mathcal{X}_3, \mathcal{Q}_3) = N_2 \bullet_{\mathrm{b}} e_l$, then $\mathcal{X}_3 = Ky + Ku + Kw$ and

$$\mathcal{Q}_3 = (y, \beta w, \beta y, \alpha u)(w, \gamma u).$$

Finally, By contracting the double link e_u on N_3, let $N_4 = (\mathcal{X}_4, \mathcal{Q}_4) = N_3 \bullet_{\mathrm{b}} e_u$, then $\mathcal{X}_4 = Ky + Kw$ and $\mathcal{Q}_4 = (y, \beta w, \beta y, \alpha w)$.

Now, the basic transformation can not be done anymore on N_4. N_4 is a map on the Klein bottle, as shown in Fig.4.7.

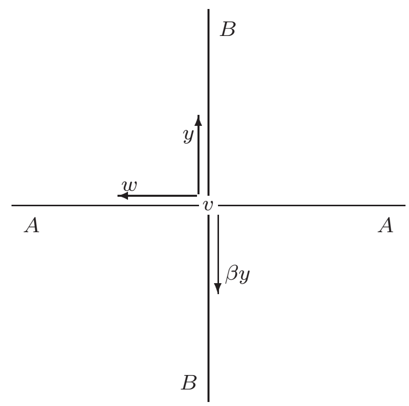


Fig.4.7 A map basic equivalent to N

From the two patterns, it is seen that $M_4 \not\sim_{\mathrm{bc}} N_4$, and hence $M \not\sim_{\mathrm{bc}} N$. Although their Euler characteristic are the same, *i.e.*,

$$\chi(M) = \chi(M_4) = 1 - 2 + 1 = \chi(N_4) = \chi(N),$$

their orientability are different.

Activities on Chapter IV

IV.5 Observations

O4.1 For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, observe how many orbits does the group $\Psi_{\{\alpha,\mathcal{P}\}}$ have on the ground set $\mathcal{X}_{\alpha,\beta}$? What condition is it for its transitivity.

O4.2 For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, observe how many orbits does the group $\Psi_{\{\beta,\mathcal{P}\gamma\}}$, $\gamma = \alpha\beta$, have on the ground set $\mathcal{X}_{\alpha,\beta}$? What condition is it for its transitivity.

O4.3 For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, observe how many orbits does the group $\Psi_{\{\beta,\mathcal{P}\}}$ have on the ground set $\mathcal{X}_{\alpha,\beta}$? What condition is it for its transitivity.

O4.4 For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, observe how many orbits does the group $\Psi_{\{\alpha,\mathcal{P}\gamma\}}$, $\gamma = \alpha\beta$, have on the ground set $\mathcal{X}_{\alpha,\beta}$? What condition is it for its transitivity.

O4.5 If map $M = (\mathcal{X}, \mathcal{P})$ is orientable, is $M - e_x$ always orientable for any e_x, $x \in \mathcal{X}$? If yes, explain the reason; otherwise, by an example.

O4.6 If map $M = (\mathcal{X}, \mathcal{P})$ is nonorientable, is $M - e_x$ always nonorientable for any e_x, $x \in \mathcal{X}$? If yes, explain the reason; otherwise, by an example.

O4.7 If map $M = (\mathcal{X}, \mathcal{P})$ is orientable, is $M \bullet e_x$ always orientable for any e_x, $x \in \mathcal{X}$? If yes, explain the reason; otherwise, by an example.

O4.8 If map $M = (\mathcal{X}, \mathcal{P})$ is nonorientable, is $M \bullet e_x$ always

nonorientable for any e_x, $x \in \mathcal{X}$? If yes, explain the reason; otherwise, by an example.

O4.9 If map $M = (\mathcal{X}, \mathcal{P})$ is orientable, is $M + e_x$ always orientable for any e_x, $x \notin \mathcal{X}$? If yes, explain the reason; otherwise, by an example.

O4.10 If map $M = (\mathcal{X}, \mathcal{P})$ is nonorientable, is $M + e_x$ always nonorientable for any e_x, $x \notin \mathcal{X}$? If yes, explain the reason; otherwise, by an example.

O4.11 If map $M = (\mathcal{X}, \mathcal{P})$ is orientable, is $M \circ e_x$ always orientable for any e_x, $x \notin \mathcal{X}$? If yes, explain the reason; otherwise, by an example.

O4.12 If map $M = (\mathcal{X}, \mathcal{P})$ is nonorientable, is $M \circ e_x$ always nonorientable for any e_x, $x \notin \mathcal{X}$? If yes, explain the reason; otherwise, by an example.

O4.13 Show by example that a face of an orientable map M does not correspond to a cocycle on its dualM^*.

IV.6 Exercises

E4.1 Try to prove Lemma4.1 in three different manners.

E4.2 For a map M, prove that there exists a nonnegative integer n and basic transformations $\pi_1, \pi_2, \cdots, \pi_n$ such that

$$M^* = \prod_{i=1}^{n} \pi_i M \tag{4.8}$$

where M^* is the dual of M.

E4.3 For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, the group $\Psi_{\{\gamma,\mathcal{P}\}}$, $\gamma = \alpha\beta$, with two orbits on $\mathcal{X}_{\alpha,\beta}$ is known. Prove that

$$\chi(M) = 0 \pmod{2}. \tag{4.9}$$

E4.4 Prove that a map$M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is nonorientable if, and

only if, there exist $x, y \in \mathcal{X}$ such that

$$|Ky \cap \{x\}_{\Psi_{\{\alpha\beta,\mathcal{P}\}}}| > 2.$$

E4.5 Try to prove Corollary 4.4 in two different manners.

E4.6 If a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ has only one face, prove that M is nonorientable if, and only if, there exists an $x \in \mathcal{X}$ such that $\alpha x \in \{x\}_{\mathcal{P}\gamma}$ where $\gamma = \alpha\beta$.

For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, let $\mathcal{A}$ be the set of all orbits of $\mathcal{P}$ on $\mathcal{X}_{\alpha,\beta}$. Graph $G_M = (V, E)$ where $V = \mathcal{A}$ and

$$E = \{(A, B)|\exists\ x \in \mathcal{X}, x \in A \text{ and } \gamma x \in B\}.$$

And, G_M is called the *subsidiary graph* of M.

E4.7 For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, prove that the group $\Psi_{\{\alpha\beta,\mathcal{P}\}}$ has two orbits on $\mathcal{X}_{\alpha,\beta}$ if, and only if, the subsidiary graph G_M of M has two connected components.

For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, $\gamma = \alpha\beta$, let $f_i = \{(x_i)_{\mathcal{P}\gamma}, (\beta x_i)_{\mathcal{P}\gamma}\}$, $i = 1, 2, \cdots, \phi$, be all the faces of M. If a set $S = \{s_i|1 \leq i \leq \phi\}$ of orbits of permutation $\mathcal{P}\gamma$ on $\mathcal{X}_{\alpha,\beta}$ satisfies $|S \cap f_i| = 1$,$i = 1, 2, \cdots, \phi$, then S ia called a *face representative* of M.

Let graph $G_S = (V, E)$ be with $V = S$ as a face representative of M and $e = (s, t) \in E$ as a pair of faces s and t with an edge in common. On E, define a weight function

$$w(e) = \begin{cases} 0, \text{ there exists } x \in s \text{ such that } \gamma x \in t; \\ 1, \text{ otherwise} \end{cases}$$

for $e = (s, t) \in E$. And, (G_S, w) is called an *associate net* of M.

On an associate net (G_S, w), $G_S = (V, E)$, of a map M, if there exists a label $l(v) \in \{0, 1\}$ for vertex $v \in V$ such that for any $e = (u, v) \in E$,

$$l(u) + l(v) = w(e) \pmod 2,$$

then the associate net (G_S, w) is said to be *balanced*.

E4.8 For a map M, prove that if one of its associate nets is balanced, then all of its associate net are balanced.

E4.9 Prove that a map is orientable if, and only if, the map has an associate net balanced.

For a graph $G = (V, E)$, $A \subseteq E$, if V has a 2-*partition*, *i.e.*, $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$, such that

$$A = \{(u, v) \in E | u \in V_1, v \in V_2\},$$

then A is called a *cocycle* of G.

E4.10 Prove that a map M is orientable if, and only if, for a face representative S of M, its associate net (G_S, w) has $\{e \in E | w(e) = 1\}$ as a cocycle.

E4.11 Prove that a map M has its Euler characteristic 2 当 if, and only if, each of its faces corresponds to a cocycle of the under graph of its dual M^*.

IV.7 Researches

R4.1 Characterize that the under graph of a map has an super map with its Euler characteristic 1.

R4.2 Characterize that the under graph of a map has an super map with its Euler characteristic 0.

R4.3 For any orientable map, characterize that the under graph of the map has an super orientable map with its Euler characteristic 0.

R4.4 For a vertex regular map and a given integer $g \leq 1$, characterize that the under graph of the map has a super map with its Euler characteristic g.

R4.5 For a vertex regular orientable map and a given integer $g \leq 0$, characterize that the under graph of the map has a super map with its Euler characteristic $2g$.

A graph which has a spanning circuit ia called a *Hamiltonian graph*. Such a spanning circuit is called a *Hamiltonian circuit* of the

graph. If a map has its under graph Hamiltonian, then it is called a *Hamiltonian map*.

R4.6 For a Hamiltonian map and a given integer $g \leq 1$, characterize that the under graph of the map has a super nonorientable map with its Euler characteristic g.

R4.7 For a Hamiltonian map and a given integer $g \leq 0$, characterize that the under graph of the map has a super orientable map with its Euler characteristic $2g$.

For a vertex 3-map(or cubic map), if it has only i-face and j-face, $i \neq j$, $i, j \geq 3$, then it is called an $(i,j)_f$-*map*.

R4.8 For a given integer $g \leq 1$, determine the number of $(3,4)_f$-map of order $n(n \geq 1)$ with Euler characteristic g.

R4.9 For a given integer $g \leq 1$, determine the number of $(4,5)_f$-map of order $n(n \geq 1)$ with Euler characteristic g.

R4.10 For a given integer $g \leq 1$, determine the number of $(5,6)_f$-map of order $n(n \geq 1)$ with Euler characteristic g.

R4.11 Given a graph G of order $n(n \geq 4)$, determine the condition for G have a super $(n-1,n)_f$-map.

On a $(n-1,n)_f$-map of order $n(n \geq 4)$, let ϕ_1 be the number of $(n-1)$-faces. If its Euler characteristic is $g \leq 1$, then n and ϕ_1 should satisfy the following condition:

$$(n-1)|(n(n-g)+\phi_1), \tag{4.10}$$

i.e., $n-1$ is a facer of $n(n-g)+\phi_1$.

R4.12 Given an integer $g \leq 1$, for any positive numbers n and ϕ_1 satisfying (4.10), determine if there exists a $(n-1,n)_f$-map with its Euler characteristic g.

Chapter V

Orientable Maps

- Any irreducible orientable map under basic subtracting edges is defined to be a butterfly. However, an equivalent class may have more than 1 butterflies.

- The simplified butterflies are for the standard orientable maps to show that each equivalent class has at most 1 simplified butterfly.

- Reduced rules are for transforming a map(unnecessary to be orientable) into another butterfly, if orientable, in the same equivalent class. A basic rule is extracted for deriving all other rules.

- Principles only for orientable maps are clarified to transform any map to a simplified butterfly in the same equivalent class. Hence, each equivalent class has at least 1 simplified butterfly.

- Orientable genus instead of the Euler characteristic is an invariant in an equivalent class to show that orientable genus itself determine the equivalent class.

V.1 Butterflies

On the basis of Chapter IV, this chapter discusses orientable maps with a standard form in each of basic equivalent classes. If an

orientable map has only one vertex and only one face, then it is called a *butterfly*.

Lemma 5.1 In each of basic equivalent classes, there exists a map with only one vertex.

Proof For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, if M has at least two vertices, from the transitive axiom, there exists an $x \in \mathcal{X}_{\alpha,\beta}$ such that $(x)_{\mathcal{P}}$ and $(\gamma x)_{\mathcal{P}}$, $\gamma = \alpha\beta$, determine two distinct vertices. Because e_x is a link, by basic contracting e_x, $M' = M \bullet_{\rm b} e_x \sim_{\rm bc} M$. Then, M' has one vertex less than M does. In view of Theorem 3.10, M' is also a map. If M' does not have only one vertex, the procedure is permitted to go on with M' instead of M. By the finite recursion principle a map M' with only one vertex can be found such that $M' \sim_{\rm bc} M$. This is the lemma. □

A map with only one vertex is also called a *single vertex map* , or in brief, a *petal bundle* .

Lemma 5.2 In a basic equivalent class of maps, there exists a map with only one face.

Proof For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, if M has at least two faces, from the transitive axiom, there exists an $x \in \mathcal{X}_{\alpha,\beta}$ such that $(x)_{\mathcal{P}\gamma}$ and $(\gamma x)_{\mathcal{P}\gamma}$, $\gamma = \alpha\beta$, determine two distinct faces. Because e_x is single, by basic deleting e_x, $M' = M -_{\rm b} e_x \sim_{\rm bc} M$. Now, M' has one face less than M does. From Theorem 3.10, M' is still a map . Thus, if M' is not with only one face, this procedure is allowed to go on with M' instead of M. By the finite recursion principle, a map M' with only one face can be finally found such that $M' \sim_{\rm bc} M$. The lemma is proved. □

In fact, on the basis of Theorem 3.6, Lemma 5.1 and Lemma 5.2 are mutually dual. Furthermore, what should be noticed is the independence of the orientability for the two lemmas.

Theorem 5.1 For any orientable map M, there exists a butterfly H such that $H \sim_{\mathrm{bc}} M$.

Proof If M has at least two vertices, from Lemma 5.1, there exists a single vertex map $L \sim_{\mathrm{bc}} M$. In virtue of Theorem 4.3, L is still orientable. if L has at least two faces, from Lemma 5.2, there exists a single face map $H \sim_{\mathrm{bc}} L$. In virtue of Theorem 4.3, H is still orientable. Since H has, finally, both one vertex and one face, H is a butterfly. Therefore, $H \sim_{\mathrm{bc}} L \sim_{\mathrm{bc}} M$. This is the theorem. □

This theorem enables us only to discuss butterflies for the Basic equivalence classes of maps without loss of generality.

V.2 Simplified butterflies

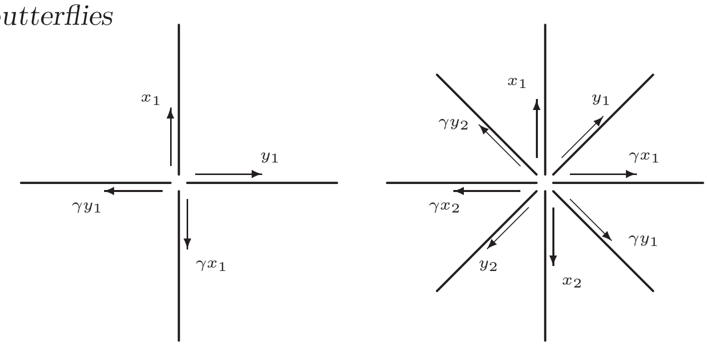
Let $O_k = (\mathcal{X}_k, \mathcal{J}_k)$, $k \geq 0$, where

$$\mathcal{X}_k = \begin{cases} \emptyset, \text{ 当 } k = 0; \\ \sum_{i=1}^{k} (Kx_i + Ky_i), \text{ 当 } k \geq 1 \end{cases} \tag{5.1}$$

and

$$\mathcal{J}_k = \begin{cases} (\emptyset), \text{ 当 } k = 0; \\ (\prod_{i=1}^{k} \langle x_i, y_i, \gamma x_i, \gamma y_i \rangle), \text{ 当 } k \geq 1. \end{cases} \tag{5.2}$$

It is east to check that all O_k, $k \geq 0$, are maps. And, they are called *O-standard maps*. When $k = 1$ and 2, O_1 and O_2 are, respectively given in (a) and (b) of Fig.5.1.



(a) O_1 (b) O_2

Fig.5.1 Two O-standard maps

Note 5.1 When $k = 0$, $O_0 = (\emptyset, \emptyset)$ is seen as the degenerate case of a map with no edge. For example, what is obtained by basic deleting an edge on $\hat{L}_0 = (Kx, (x, \gamma x))$ is just O_0. Usually, it is seen as the map with only one vertex without edge, or called the *trivial map*.

Lemma 5.3 For any $k \geq 0$, O-standard map O_k is orientable.

Proof When $k = 0$, from $O_0 = (Kx, (x, \gamma x)) -_{\mathrm{b}} e_x$,

$$O_0 \sim_{\mathrm{bc}} (Kx, (x, \gamma x)).$$

Because $\{x\}_{\Psi_{\{(x,\gamma x),\gamma\}}} = \{x, \gamma x\}$ and $\{\alpha x\}_{\Psi_{\{(x,\gamma x),\gamma\}}} = \{\alpha x, \beta x\}$ are two orbits, $(Kx, (x, \gamma x))$ is orientable. In view of Theorem 4.3, O_0 is orientable.

For $k \geq 1$, assume, by induction, that $O_{k-1} = (\mathcal{X}_{k-1}, \mathcal{J}_{k-1})$ is orientable. From (5.1) and (5.2), group $\Psi_{\{\mathcal{J}_{k-1},\gamma\}}$ has two orbits as

$$\{x_1\}_{\Psi_{\{\mathcal{J}_{k-1},\gamma\}}} = \{x_i, \gamma x_i | 1 \leq i \leq k-1\}$$

and

$$\{\alpha x_1\}_{\Psi_{\{\mathcal{J}_{k-1},\gamma\}}} = \{\alpha x_i, \beta x_i | 1 \leq i \leq k-1\}.$$

For $O_k = (\mathcal{X}_k, \mathcal{J}_k)$, from (5.2), $\mathcal{J}_k = (\langle \mathcal{J}_{k-1} \rangle, x_k, y_k, \gamma x_k, \gamma y_k)$. Group $\Psi_{\{\mathcal{J}_k,\gamma\}}$ has only two orbits as

$$\begin{aligned}\{x_1\}_{\Psi_{\{\mathcal{J}_k,\gamma\}}} &= \{x_1\}_{\Psi_{\{\mathcal{J}_{k-1},\gamma\}}} \cup \{x_k, y_k, \gamma x_k, \gamma y_k\} \\ &= \{x_i, \gamma x_i | 1 \leq i \leq k-1\} \cup \{x_k, y_k, \gamma x_k, \gamma y_k\} \\ &= \{x_i, \gamma x_i | 1 \leq i \leq k\}\end{aligned}$$

and

$$\begin{aligned}\{\alpha x_1\}_{\Psi_{\{\mathcal{J}_k,\gamma\}}} &= \{\alpha x_1\}_{\Psi_{\{\mathcal{J}_{k-1},\gamma\}}} \cup \{\alpha x_k, \alpha y_k, \beta x_k, \beta y_k\}\\ &= \{\alpha x_i, \beta x_i | 1 \le i \le k-1\} \cup \{\alpha x_k, \alpha y_k, \beta x_k, \beta y_k\}\\ &= \{\alpha x_i, \beta x_i | 1 \le i \le k\}.\end{aligned}$$

Therefore, O_k, $k \ge 1$, are all orientable. $\square$

Lemma 5.4 For any $k \ge 0$, O-standard map O_k has only one face.

Proof When $k = 0$, since $O_0 = (Kx, (x, \gamma x)) -_{\mathrm{b}} e_x$, O_0 should have one face which is composed from the two faces (x) and (αx) of $(Kx, (x, \gamma x))$. Therefore, O_0 has only one face(seen as a degenerate case because of no edge).

For any $O_k = (\mathcal{X}_k, \mathcal{J}_k)$, $k \ge 1$, from (5.1) and (5.2),

$$(x_1)_{\mathcal{J}_k\gamma} = (x_1, \gamma y_1, \gamma x_1, y_1, \cdots, x_k, \gamma y_k, \gamma x_k, y_k)$$

is a face of O_k. However, since

$$|\{x_1\}_{\mathcal{J}_k\gamma}| = \frac{1}{2}|\mathcal{X}_k|,$$

O_k has only this face. $\square$

From (5.2), each O-standard map has only one vertex (O_0 is the degenerate case of no incident edge). Based on the above two lemmas, any O-standard map is a butterfly. Because of the simplicity in form for them, they are called *simplified butterflies* . Since for any $k \ge 0$, simplified butterfly O_k has $2k$ edges, one vertex and one face, its Euler characteristic is

$$\chi(O_k) = 2 - 2k. \tag{5.3}$$

Lemma 5.5 For any two simplified butterflies O_i and O_j, $i, j \ge 0$, $O_i \sim_{\mathrm{bc}} O_j$ if, and only if, $i = j$.

Proof Because the sufficiency, *i.e.*, the former $O_i \sim_{\mathrm{el}} O_j$ is derived from the latter $i = j$, is natural, only necessary to prove the necessity.

By contradiction. Suppose $i \neq j$, but $O_i \sim_{\text{bc}} O_j$. Because of the basic equivalence, from Theorem 4.5,

$$\chi(O_i) = \chi(O_j).$$

However, from (5.3) and the condition $i \neq j$,

$$\chi(O_i) = 2 - 2i \neq 2 - 2j = \chi(O_j).$$

This is a contradiction. □

Theorem 5.2 In each of the basic equivalent classes of orientable maps, there is at most one map which is a simplified butterfly.

Proof By contradiction. Suppose simplified butterflies O_i and O_j, $i \neq j$, $i, j \geq 0$, are in the same class. However, this is a contradiction to Lemma 5.5. □

In the next two sections of this chapter, it will be seen that in each basic equivalent class of orientable maps, there is at least one map which is a simplified butterfly.

On the basis of Theorem 4.5, two butterflies of the same size have the same Euler characteristic. Do they all simplified butterflies? However, the answer is negative!

Example 5.1 Observe map $M = (\mathcal{X}, \mathcal{J})$ where

$$\mathcal{X} = Kx_1 + Ky_1 + Kx_2 + Ky_2$$

and

$$\mathcal{J} = (x_1, y_1, x_2, y_2, \gamma x_1, \gamma y_1, \gamma x_2, \gamma y_2).$$

Because the face

$$(x_1)_{\mathcal{J}\gamma} = (x_1, \gamma y_1, x_2, \gamma y_2, \gamma x_1, y_1, \gamma x_2, y_2)$$

has 8 elements, half the elements of ground set, M has only one face. Hence, M is a butterfly, but not a simplified butterfly. Actually, the simplified butterfly with the sane Euler characteristic of M is O_2.

V.3 Reduced rules

Although butterflies are necessary to find a representative for each basic equivalent class of orientable maps, single vertex maps are restricted in this section for such a classification based on Lemma 5.1.

For convenience, the basic equivalence between two maps are not distinguished from that between their basic permutations. In other words, $(\mathcal{X}_1, \mathcal{P}_1) \sim_{\rm bc} (\mathcal{X}_2, \mathcal{P}_2)$ stands for $\mathcal{P}_1 \sim_{\rm bc} \mathcal{P}_2$.

Lemma 5.6 For a single vertex map $M = (\mathcal{X}, \mathcal{J})$, if

$$\mathcal{J} = (R, x, \gamma x, S)$$

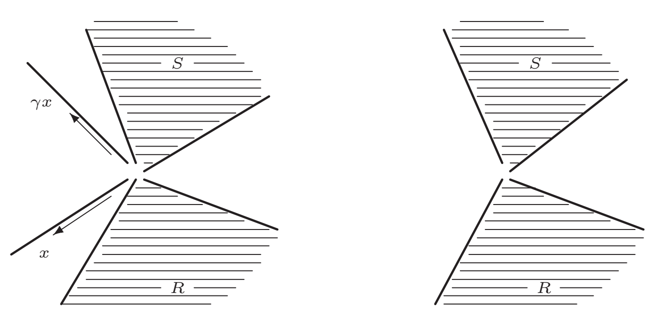
where R and S are two linear orders on $\mathcal{X}$, then

$$\mathcal{J} \sim_{\rm bc} (R, S), \tag{5.4}$$

as shown from (a) to (b) in Fig.5.2.

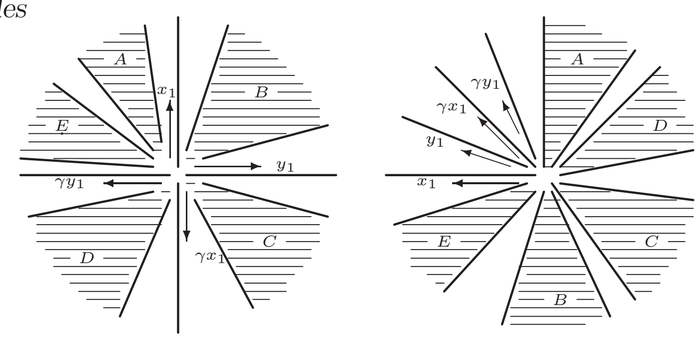
Proof Because $\mathcal{J} = (R, x, \gamma x, S)$, $(\mathcal{J}\gamma)\gamma x = \mathcal{J}x = \gamma x$, *i.e.*, $(\gamma x)_{\mathcal{J}\gamma} = (\gamma x)$ is a face. Because e_x is a single edge, by basic deleting e_x on M, $M' = M -_{\rm b} e_x = (\mathcal{X} - Kx, \mathcal{J}')$, $\mathcal{J}' = (R, S)$. From $M \sim_{\rm bc} M'$, $\mathcal{J} \sim_{\rm bc} \mathcal{J}' = (R, S)$. $\square$

This lemma enables us to transform a single vertex map into another single vertex map with one face less in a basic equivalent class.



(a) (b)

Fig.5.2 Reduced rule (5.4)



(a) (b)

Fig.5.3 Reduced rule (5.5)

Lemma 5.7 For $(\mathcal{X}_{\alpha,\beta}, \mathcal{J})$, if $\mathcal{J} = (A, x, B, y, C, \gamma x, D, \gamma y, E)$ where A, B, C, D and E are all linear orders on $\mathcal{X}$, then

$$\mathcal{J} \sim_{\rm bc} (A, D, C, B, E, x, y, \gamma x, \gamma y), \tag{5.5}$$

as shown from (a) to (b) in Fig.5.3.

Proof Four steps are considered for each step as a claim.

Claim 1 $\mathcal{J} \sim_{\rm bc} (E, A, x, z, D, C, \gamma x, \gamma z, B)$.

Proof For the angle pair $(\alpha x, \mathcal{J}x)$ and $(\beta x, \mathcal{J}\gamma x)$ of

$$\mathcal{J} = (A, x, B, y, C, \gamma x, D, \gamma y, E),$$

by basic splitting e_z(a link), get

$$\mathcal{J} \sim_{\rm bc} \mathcal{J}_1 = (D, \gamma y, E, A, x, z)(\gamma z, B, y, C, \gamma x).$$

Then, since e_y is a link, by basic contracting e_y on $\mathcal{J}_1$, get

$$\mathcal{J}_1 \sim_{\rm bc} \mathcal{J}_2 = (E, A, x, z, D, C, \gamma x, \gamma z, B).$$

This is the conclusion of Claim 1. Here, $\mathcal{J}_1$ and $\mathcal{J}_2$ are, respectively, shown in (a) and (b) of Fig.5.4.

Claim 2 $\mathcal{J}_2 \sim_{\rm bc} (y, A, x, B, E, \gamma y, D, C, \gamma x)$.

Proof For the pair of angle $\langle \alpha z, \mathcal{J}_2 z\rangle$ and angle between E and A on $\mathcal{J}_2 = (E, A, x, z, D, C, \gamma x, \gamma z, B)$, by basic splitting e_y(a link), get $\mathcal{J}_2 \sim_{\rm bc} \mathcal{J}_3 = (A, x, z, y)(\gamma y, D, C, \gamma x, \gamma z, B, E)$. Then, since e_z is a link, by basic contracting e_z on $\mathcal{J}_3$, get

$$\mathcal{J}_3 \sim_{\rm bc} \mathcal{J}_4 = (y, A, x, B, E, \gamma y, D, C, \gamma x).$$

This is the conclusion of Claim 2. Here, $\mathcal{J}_3$ and $\mathcal{J}_4$ are, respectively, shown in (a) and (b) of Fig.5.5.

Claim 3 $\mathcal{J}_4 \sim_{\mathrm{bc}} (B, E, z, A, y, \gamma z, \gamma y, D, C)$.

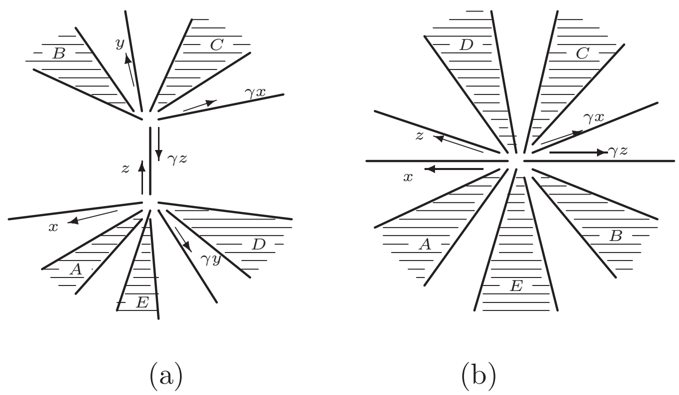


Fig.5.4 For Claim 1

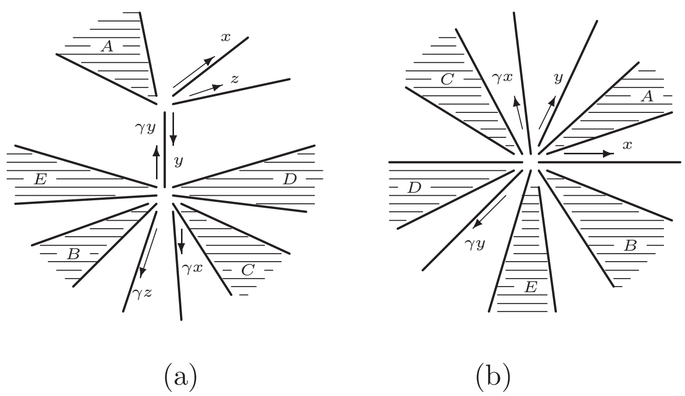


Fig.5.5 For Claim 2

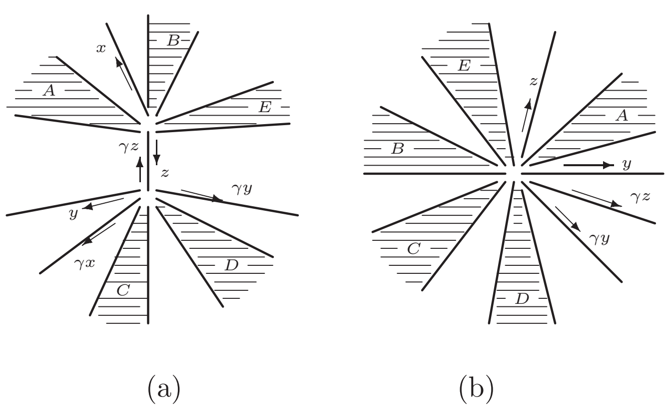


Fig.5.6 For Claim 3

Proof For the angle pair $(\alpha y, \mathcal{J}_4 y)$ and $(\alpha \mathcal{J}_4^{-1} \gamma y, \gamma y)$ on $\mathcal{J}_4 =$

$(y, A, x, B, E, \gamma y, D, C, \gamma x)$, by basic splitting e_z(a link), get

$$\mathcal{J}_4 \sim_{\rm bc} \mathcal{J}_5 = (A, x, B, E, z)(\gamma z, \gamma y, D, C, \gamma x, y).$$

Then, since e_x is a link, by basic contracting e_x on $\mathcal{J}_3$, get

$$\mathcal{J}_5 \sim_{\rm bc} \mathcal{J}_6 = (B, E, z, A, y, \gamma z, \gamma y, D, C).$$

This is the conclusion of Claim 3. Here, $\mathcal{J}_5$ and $\mathcal{J}_6$ are, respectively, shown in (a) and (b) of Fig.5.6.

Claim 4 $\mathcal{J}_6 \sim_{\rm bc} (A, D, C, B, E, z, x, \gamma z, \gamma x)$.

Proof For the angle pair $(\alpha z, \mathcal{J}_6 z)$ and $(\alpha y, \gamma z)$ of

$$\mathcal{J}_6 = (B, E, z, A, y, \gamma z, \gamma y, D, C),$$

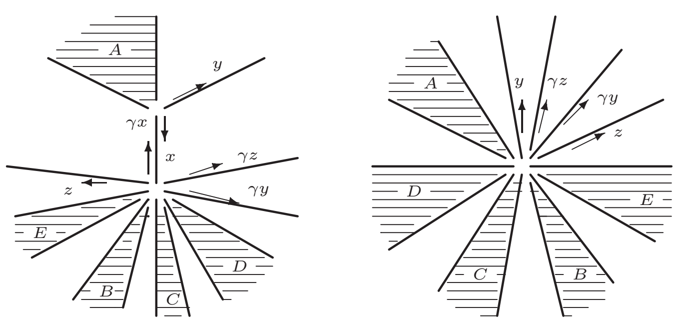
by basic splitting e_x (a link), get

$$\mathcal{J}_6 \sim_{\rm bc} \mathcal{J}_7 = (A, y, x)(\gamma x, \gamma z, \gamma y, D, C, B, E, z).$$

Then, since e_y is a link, by basic contracting e_y on $\mathcal{J}_7$, get

$$\mathcal{J}_7 \sim_{\rm bc} \mathcal{J}_8 = (A, D, C, B, E, z, x, \gamma z, \gamma x).$$

This is the conclusion of Claim 4. Here, $\mathcal{J}_7$ and $\mathcal{J}_8$ are, respectively, shown in (a) and (b) of Fig.5.7.



(g) (h)

Fig.5.7 Claim 4

On the basis of the four claims above,

$$\begin{aligned}\mathcal{J} &\sim_{\rm bc} \mathcal{J}_2 \sim_{\rm bc} \mathcal{J}_4 \sim_{\rm bc} \mathcal{J}_6 \sim_{\rm bc} \mathcal{J}_8 \\ &= (A, D, C, B, E, x, y, \gamma x, \gamma y).\end{aligned}$$

This is (5.5). $\square$

An attention should be paid to that Lemma 5.6 and Lemma 5.7 are both valid for orientable and nonorientable maps. They are called *reduced rules* for maps. More precisely, They are explained as in the following.

Reduced rule 1 A map with its basic permutation $\mathcal{J}$ is basic equivalent to what is obtained by leaving off such a successive elements $\langle x, \gamma x\rangle$, $x \in \mathcal{X}$, on $\mathcal{J}$.

Reduced rule 2 A map with its basic permutation $\mathcal{J}$ in the form as $(A, x.B,\ y, C, \gamma x, D, \gamma y, E)$ is basic equivalent to what is obtained by interchanging the linear order B between x and y and the linear order D between γx and γy, and then leaving off $x, y, \gamma x$ and γy and putting $\langle x, y, \gamma x, \gamma y\rangle$ behind E on $\mathcal{J}$.

V.4 Orientable principles

This section is centralized on discussing the basic equivalent classes of orientable maps. The main purpose is to extract that there is at least one simplified butterfly in each class. From the first section of this chapter, it is known that each class is considered for only butterflies without loss of generality.

Theorem 5.3 For a butterfly $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{J})$, $\gamma = \alpha\beta$, if no $x, y \in \mathcal{X}$ exist such that $\mathcal{J} = (A, x, B, y, C, \gamma x, D, \gamma y, E)$ where A, B, C, D and E are all linear orders on $\mathcal{X}_{\alpha,\beta}$, then

$$M \sim_{\mathrm{bc}} O_0, \tag{5.6}$$

i.e., the trivial map(the degenerate simplified butterfly without edge).

Proof For convenience, in a cyclic permutation $\mathcal{J}$ on $\mathcal{X}_{\alpha,\beta}$, if two elements $x, y \in \mathcal{X}_{\alpha,\beta}$ are in the form as $\mathcal{J} = (A, x, B, y, C, \gamma x, D, \gamma y, E)$, then they are said to be *interlaced*; otherwise, *parallel*.

Claim If any two elements are parallel on $\mathcal{J}$, then there is an element $x \in \mathcal{X}_{\alpha,\beta}$ such that $\langle x, \gamma x\rangle \subseteq \mathcal{J}$, *i.e.*, $\langle x, \gamma x\rangle$ is a segment of $\mathcal{J}$ itself.

Proof By contradiction. If no such an elements exists on $\mathcal{X}_{\alpha,be}$, then for any $x_1 \in \mathcal{X}$, there is a nonempty linear order B_1 on $\mathcal{X}$ such that $\mathcal{J} = (A_1, x, B_1, \gamma x, C_1)$ where A_1 and C_1 are some linear orders on $\mathcal{X}_{\alpha,\beta}$. Because $B_1 \neq \emptyset$, for any $x_2 \in B_1$, on the basis of orientability and x_2 and x_1 parallel, the only possibility is $\gamma x_2 \in B_1$. From the known condition, there is also a linear order $B_2 \neq \emptyset$ on $\mathcal{X}_{\alpha,\beta}$ such that $B_1 = \langle A_2, x_2, B_2, \gamma x_2, C_2\rangle$ where A_2 and C_2 are segments on B_1, *i.e.*, some linear orders on $\mathcal{X}$. Such a procedure can only go on to the infinity. This is a contradiction to the finiteness of $\mathcal{X}_{\alpha,\beta}$. Hence, the claim is true.

If $\mathcal{J} \neq \emptyset$, then from the claim, there exists an element x in $\mathcal{J}$ such that $\mathcal{J} = (A, x, \gamma x, B)$. However, because $(\gamma x)_{\mathcal{J}\gamma} = (\gamma x)$ is a face in its own right, $\mathcal{J}$ has to be with at least two faces. This is a contradiction to that $M = (\mathcal{X}, \mathcal{J})$ is a butterfly. Hence, the only possibility is $\mathcal{J} = \emptyset$, *i.e.*, (5.6) holds. □

Actually, this theorem including the claim in its proof is valid for any orientable single vertex map. Therefore, it can be seen that the reduced rule(Lemma 5.6) and the following corollary are valid for any map (orientable or nonorientable) as well.

Corollary 5.1 Let $S = \langle A, x, \gamma x, B\rangle$ be a segment on a vertex of a map M. And let M' be obtained from M by substituting $\langle A, B\rangle$ for S and afterward deleting Kx from the ground set. Then, M' is a map. And, $M' \sim_{\mathrm{bc}} M$.

Proof Because it is easy to check that $M' = M -_{\mathrm{b}} M$, from Theorem 3.10, M' is a map. This is the first statement. In view of basic deletion of an edge as a basic transformation, $M' \sim_{\mathrm{bc}} M$. □

Corollary 5.2 Let S be a segment at a vertex of a map M. If for each element x in S, γx is also in S and any two elements in

S are not interlaced, then there exists an element y in S such that $S = \langle A, y, \gamma y, B\rangle$.

Proof In the same way of proving Theorem 5.3, the conclusion is soon obtained. $\square$

Theorem 5.4 In a butterfly $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{J})$, if there are $x, y \in \mathcal{X}_{\alpha,\beta}$ such that $\mathcal{J} = (A, x, B, y, C, \gamma x, D, \gamma y, E)$, then there is an integer $k \geq 1$ such that

$$M \sim_{\rm bc} O_k, \tag{5.7}$$

i.e., the simplified butterfly with $2k$ edges.

Proof Based on Reduced rule 2(Lemma 5.7),

$$\mathcal{J} \sim_{\rm bc} (A, D, C, B, E, x, y, \gamma x, \gamma y).$$

Let $H = \langle A, D, C, B, E\rangle$. From Corollary 5.1, assume H is not in the form as S without loss of generality. From Corollary 5.2 and Theorem 5.3, H has two possibilities: $H = \emptyset$, or there exist two elements x_1 and y_1 interlaced in H.

If the former, then $\mathcal{J} \sim_{\rm bc} (x, y, \gamma x, \gamma y)$, *i.e.* , $M \sim_{\rm el} O_1$. Otherwise, *i.e.*, the latter, then $\mathcal{J} = (A_1, x_1, B_1, y_1, C_1, \gamma x_1, D_1, \gamma y_1, E_1)$. An attention should be paid to that $E_1 = \langle F_1, x, y, \gamma x, \gamma y\rangle$. In this case, from Lemma 5.7,

$$\begin{aligned}\mathcal{J} &\sim_{\rm bc} (A_1, D_1, C_1, B_1, E_1, x_1, y_1, \gamma x_1, \gamma y_1)\\ &= (A_1, D_1, C_1, B_1, F_1, x, y, \gamma x, \gamma y, x_1, y_1, \gamma x_1, \gamma y_1).\end{aligned}$$

Let $H_1 = \langle A_1, D_1, C_1, B_1, F_1\rangle$, then for H_1 instead of H, go on the procedure. According to the principle of finite recursion, it is only possible to exists an integer $k \geq 1$ such that (5.7) holds. $\square$

This theorem shows that each basic equivalent class of orientable maps has at least one map which is a simplified butterfly.

By considering Theorem 5.2, each basic equivalent class of orientable maps has, and only has, an integer $k \geq 0$ such that the simplified butterfly of size k is in the class

V.5 Orientable genus

Although Euler characteristic of a map is an invariant for basic transformation, a basic equivalent class of maps can not be determined by itself. This is shown from the map M in Example 4.1 of section IV.4 and the map N in Example 4.2 of section IV.4. They both have the same Euler characteristic. However, they are not in the same basic equivalent class of maps because M is orientable and N is nonorientable.

Now, a further invariant should be considered for a class of orientable maps under the basic equivalence.

Theorem 5.5 For any orientable map $M = (\mathcal{X}, \mathcal{P})$, there has, and only has, an integer $k \geq 0$, such that its Euler characteristic is

$$\chi(M) = 2 - 2k. \tag{5.8}$$

Proof From Theorem 5.1 and Theorem 4.5, only necessary to discuss butterflies. From Theorem 5.2 and Theorem 5.4, M has, and only has, an integer $k \geq 0$, such that $M \sim_{\mathrm{bc}} O_k$. Therefore, from (5.3),

$$\chi(M) = 2 - 2k.$$

This is (5.8). □

Corollary 5.3 The Euler characteristic of an orientable map $M = (\mathcal{X}, \mathcal{P})$ is always an even number, *i.e.*,

$$\chi(M) = 0 \pmod{2}. \tag{5.9}$$

Proof A direct result of Theorem 5.5. □

Since Euler characteristic is an invariant of a basic equivalent class, the integer k in (5.8) is an invariant as well. From Theorem 5.5, k determines a basic equivalent class for orientable maps. Since each orientable map in the basic equivalent class determined by k can be seen as an embedding of its under graph on the orientable surface of

genus k, k is also called the *genus*, or more precisely, *orientable genus* of the map. Of course, only an orientable map has the orientable genus.

From what has been discussed above, it is seen that although Euler characteristic can not determine the basic equivalent class for all maps, the Euler characteristic can certainly determine the basic equivalent class for orientable maps.

Activities on Chapter V

V.6 Observations

O5.1 Think, is there a butterfly which has 3 edges? Further, is there a butterfly with some odd number of edges? If yes, provide an example. Otherwise, explain the reason.

O5.2 Observe that any butterfly with 2 edges is a simplified butterfly. Explain the reason.

O5.3 Provide a butterfly of at most 5 edges , which is not a simplified butterfly.

O5.4 Observe that is there a superfluous operation among the four operations: basic deleting, basic appending, basic contracting and basic splitting an edge in the basic transformation for the basic equivalence? If no, indicate the role of each of them. If yes, indicate the superfluous operation with the why.

O5.5 Think, do some three of the four operations: basic deleting, basic appending, basic contracting and basic splitting an edge in the basic transformation determine an equivalence? If do, provide an example. Otherwise, explain the reason.

O5.6 Among the eight operations of the above four with increasing duplition, decreasing duplition, increasing subdivision and decreasing subdivision, how many groups of these operations are there for determining the basic equivalence.

O5.7 Can an equivalence be determined by basic deleting and basic appending an edge? If yes, observe some of invariants under the

equivalence. Otherwise, explain the reason.

O5.8 Can an equivalence be determined by basic contracting and basic splitting an edge? If yes, observe some of invariants under the equivalence. Otherwise, explain the reason.

O5.9 Can an equivalence be determined by basic increasing duplition and decreasing duplition of an edge? If yes, observe some of invariants under the equivalence. Otherwise, explain the reason.

O5.10 Can an equivalence be determined by basic increasing subdivision and decreasing subdivision of an edge? If yes, observe some of invariants under the equivalence. Otherwise, explain the reason.

O5.11 Can the procedure of proving Lemma 5.7 by four steps be improved to that by three steps? If yes, provide a proof of three steps. Otherwise, explain the why.

V.7 Exercises

E5.1 By basic deleting and basic appending an edge only, prove

$$(A, x, B, y, C, \gamma x, D, \gamma y, E) \sim_{\mathrm{bc}} (A, D, C, B, E, x, y, \gamma x, \gamma y).$$

E5.2 Prove that a map has its orientable genus 1 if, and only if,

$$M \sim_{\mathrm{bc}} (x, y, z, \gamma y)(\beta z, \beta t, \beta x, \alpha t).$$

E5.3 Provide two maps of order 3 with orientable genus 1. And, explain they are distinct.

For a set of operations $\mathcal{A}$ and a set of maps $\mathcal{B}$ not necessary to be closed under $\mathcal{A}$, if for a map $M \in \mathcal{B}$ there is no such a map $N \in \mathcal{B}$ of size less than the size of M that N can be obtained from M by operations in $\mathcal{A}$, then M is called *irreducible* under $\mathcal{A}$.

E5.4 For an integer $k \geq 0$, determine all the irreducible single vertex maps of orientable genus k under basic deleting and basic appending an edge.

E5.5 For an integer $n \geq 1$, determine all the irreducible orientable single vertex maps of size k under basic deleting and basic appending an edge.

E5.6 For an integer $n \geq 1$, determine all the irreducible orientable single vertex maps of size k under basic contracting and basic splitting an edge.

E5.7 For an integer $k \geq 0$, determine all the irreducible single vertex maps of orientable genus k under basic contracting and basic splitting an edge.

E5.8 Prove that a complete graph K_n of order $n \geq 3$ has an orientable single face embedding if and only if, $\binom{n-1}{2}$ is even.

E5.9 Prove that a complete graph K_n of order $n \geq 3$ has an orientable two face embedding if and only if, $\binom{n-1}{2}$ is odd.

E5.10 Prove that a complete bipartite graph $K_{m,n}$ of order $n+m \geq 3$ has an orientable single face embedding if, and only if, $(m-1)(n-1)$ is even.

E5.11 Prove that a complete bipartite graph $K_{m,n}$ of order $n+m \geq 3$ has an orientable two face embedding if, and only if, $(m-1)(n-1)$ is odd.

An *n-cube* is the graph formed by the skeleton of the n-dimensional cuboid. The order of an n-cube is 2^n and the size, $n2^{n-1}$.

E5.12 Prove that any n-cube, $n \geq 2$, has an two face embedding.

A map of orientable genus 0 is also said to be *planar*.

E5.13 Prove that a map M is planar if, and only if, each face of M corresponds to a cocycle(See 1.6) in the under graph $G(M^*)$ of its dual M^*.

E5.14 Prove that a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is orientable if, and

only if, $\mathcal{P}$ can be transformed into cyclic permutation $\mathcal{J}$ such that

$$\mathcal{J} = (\prod_{i=1}^{k}\langle x_i, y_i\rangle, \prod_{i=1}^{k}\langle \gamma x_i, \gamma y_i\rangle). \tag{5.10}$$

V.8 Researches

If the travel formed by a face in a map can be partitioned into *tours* (travel without edge repetition), then the face is said to be *pan-tour*. A map with all of its faces pan-tour is called a *pan-tour map* . Because any tour can be partitioned into circuits, a pan-tour map is, in fact, a favorable map as mentioned in 2.8. A pre-proper embedding corresponds to what is called a *tour map* because each face forms a tour in its under graph.

R5.1 Characterize and recognize that a graph has a super map which is an orientable pan-tour map.

R5.2 Characterize and recognize that a graph has a super map which is an orientable tour map.

R5.3 *Orientable pan-tour conjecture*. Prove, or improve, that any nonseparable graph has a super map which is an orientable pan-tour map.

R5.4 *Orientable tour conjecture*. Prove, or improve, that any nonseparable graph has a super map which is an orientable tour map.

R5.5 Characterize and recognize that a graph has a super map which is an orientable pre-proper map.

R5.6 *Orientable proper map conjecture*. Prove, or improve, that any nonseparable graph has a super map which is an orientable proper map.

The *orientable minimum pan-tour genus* , usually called *orientable pan-tour genus*, of a graph is the minimum among all orientable genera of its super pan-tour maps. Similarly, the *orientable pan-tour*

maximum genus of a graph is the maximum among all orientable genera of its super pan-tour maps.

R5.7 Determine the orientable maximum pan-tour genus of a graph.

R5.8 Determine the orientable maximum tour genus of a graph.

R5.9 Determine the orientable pan-tour genus of a graph.

R5.10 Determine the orientable tour genus of a graph.

Although many progresses has been made on determining the orientable maximum genus of a graph, the study on determining orientable (maximum) pan-tour genus, or orientable (maximum) tour genus of a graph does not lead to any notable result yet. This suggests to investigate their bounds(upper or lower) for some class of graphs.

R5.11 Characterize the class of graphs in which each graph has its orientable maximum pan-tour genus equal to its orientable maximum genus. Find the least upper bound of the absolute difference between the orientable maximum pan-tour genus and the orientable maximum genus for a class of graphs with the two genera not equal.

R5.12 Characterize the class of graphs in which each graph has its orientable maximum tour genus equal to its orientable maximum genus. Find the least upper bound of the absolute difference between the orientable maximum pan-tour genus and the orientable maximum genus for a class of graphs with the two genera not equal.

R5.13 Characterize the class of graphs in which each graph has its orientable pan-tour genus equal to its orientable genus. Find the least upper bound of the absolute difference between the orientable pan-tour genus and the orientable genus for a class of graphs with the two genera not equal.

R5.14 Characterize the class of graphs in which each graph has its orientable tour genus equal to its orientable genus. Find the least upper bound of the absolute difference between the orientable

tour genus and the orientable genus for a class of graphs with the two genera not equal.

Chapter VI

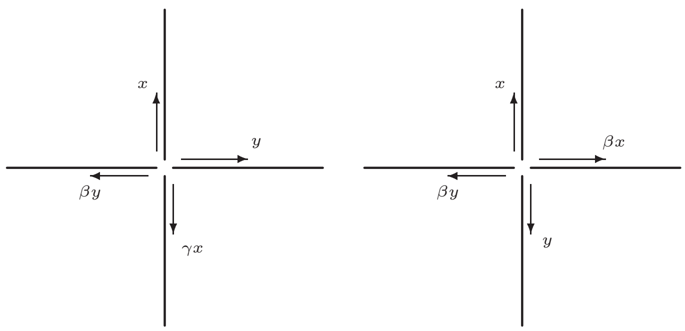
Nonorientable Maps

- Any irreducible nonorientable map under basic subtraction of an edge is defined to be a barfly. However, an equivalent class may have more than 1 barflies.
- The simplified barflies are for standard nonorientable maps to show that each equivalent class has at most 1 simplified barfly.
- Nonorientable rules are for transforming a barfly into another barfly in the same equivalent class. A basic rule is extracted for deriving from one to all others.
- Principles only for nonorientable maps are clarified to transform any nonorientable map to a simplified barfly in the same equivalent class. Hence, each equivalent class has at least 1 simplified barfly.
- Nonorientable genus instead of the Euler characteristic is an invariant in an equivalent class to show that nonorientable genus itself determines the equivalent class.

VI.1 Barflies

This chapter concentrate on discussing the basic equivalent classes of nonorientable maps by extracting a representative for each class. On the basis of Lemma 5.1 and Lemma 5.2, Only maps with a single

vertex and a single face are considered for this purpose without loss of generality. A nonorientable map with both a single vertex and a single face is called a *barfly*. The barfly with only one edge is the map consisted of a single twist loop, *i.e.*, $N^{(1)} = (Kx, (x, \beta x))$.



(a) $N_1^{(2)}$ (b) $N_2^{(2)}$

Fig.6.1 Barflies with two edges

Example 6.1 Two barflies of size two. Let $N_1^{(2)} = (Kx + Ky, \mathcal{I}_1)$ and $N_2^{(2)} = (Kx + Ky, \mathcal{I}_2)$(shown, respectively, in (a) and (b) of Fig.6.1) where

$$\mathcal{I}_1 = (, x, y, \gamma x, \beta y),\ \mathcal{I}_2 = (x, \beta x, y, \beta y).$$

Because of

$$(x)_{\mathcal{I}_1\gamma} = (x, \beta y, \alpha x, \alpha y)$$

and

$$(x)_{\mathcal{I}_2\gamma} = (x, \alpha x, y, \alpha y),$$

each of $N_1^{(2)}$ and $N_2^{(2)}$ has exactly one face. And, since

$$(x)_{\Psi_{\{\mathcal{I}_1,\gamma\}}} = (x)_{\Psi_{\{\mathcal{I}_2,\gamma\}}} = Kx + Ky,$$

they are both nonorientable.

As mentioned in the last section, for convenience, the scope of maps considered here for the specific purpose should be enlarged to all nonorientable one vertex maps from barflies.

Lemma 6.1 For a single vertex map $M = (\mathcal{X}_{\alpha,.be}, \mathcal{P})$, M is nonorientable if, and only if, there exists an $x \in \mathcal{X}_{\alpha,\beta}$ such that $\beta x \in \{x\}_{\mathcal{P}}$.

Proof Necessity. By contradiction. Assume that for any $x \in \mathcal{X}_{\alpha,\beta}$, there always has $\gamma x \in \{x\}_{\mathcal{P}}$, $\gamma = \alpha\beta$, then $\alpha x \notin \{x\}_{\Psi_{\{\mathcal{P},\gamma\}}}$. From 4.1, M is not nonorientable.

Sufficiency. Since $x \in \mathcal{X}_{\alpha,\beta}$ and $\beta x \in \{x\}_{\mathcal{P}}$, from Corollary 4.1 and only one vertex, $\Psi_{\{\mathcal{P},\gamma\}}$ has only one orbit on $\mathcal{X}$. Hence, M is nonorientable. $\square$

This lemma can easily be employed for checking the nonorientability of a one vertex map.

Example 6.2 Six barflies of three edges. Let $N_i^{(3)} = (Kx + Ky + Kz, \mathcal{I}_i)$, $i = 1, 2, \cdots, 6$(shown, respectively, in (a,b,$\cdots$,f) of Fig.6.2) where

$$\begin{aligned}
\mathcal{I}_1 &= (x, \beta x, y, \beta y, z, \beta z),\\
\mathcal{I}_2 &= (x, y, z, \beta y, \beta x, \beta z),\\
\mathcal{I}_3 &= (x, y, z, \beta z, \gamma x, \beta y),\\
\mathcal{I}_4 &= (x, \beta x, y, z, \beta z, \beta y),\\
\mathcal{I}_5 &= (x, y, \beta x, z, \beta y, \beta z),\\
\mathcal{I}_6 &= (x, y, z, \beta x, \gamma y, \gamma z).
\end{aligned}$$

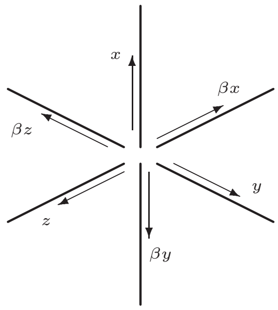
Because $\beta x \in \{x\}_{\mathcal{I}_i} \subseteq \{x\}_{\Psi_{\{\mathcal{I}_i,\gamma\}}}$, $\gamma = \alpha\beta$, $i = 1, 2, \cdots, 6$, from Lemma 6.1, they are all nonorientable. Since

$$\begin{aligned}
(x)_{\mathcal{I}_1\gamma} &= (x, \alpha x, y, \alpha y, z, \alpha z),\\
(x)_{\mathcal{I}_2\gamma} &= (x, \gamma y, z, \gamma x, y, \alpha z),\\
(x)_{\mathcal{I}_3\gamma} &= (x, \beta y, \alpha x, \gamma z, \beta z, \alpha y),\\
(x)_{\mathcal{I}_4\gamma} &= (x, \alpha x, y, \gamma z, \beta z, \alpha y),\\
(x)_{\mathcal{I}_5\gamma} &= (x, \alpha y, \beta z, , \gamma x, y, \alpha z),\\
(x)_{\mathcal{I}_6\gamma} &= (x, \alpha z, \beta y, \alpha x, \gamma y, z),
\end{aligned}$$

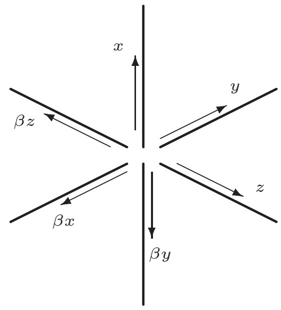
the maps $(Kx + Ky + Kz, \mathcal{I}_i)$, $i = 1, 2, \cdots, 6$, are all with only one face. Therefore, they are all barflies.

Theorem 6.1 For any nonorientable map M, there exists a barfly N such that

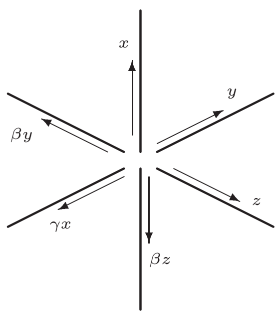
$$M \sim_{\mathrm{bc}} N. \tag{6.1}$$



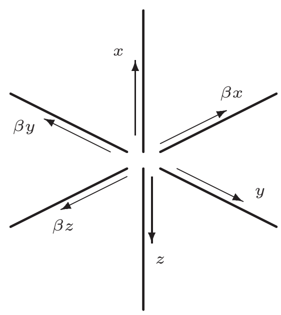
(a) $N_1^{(3)}$



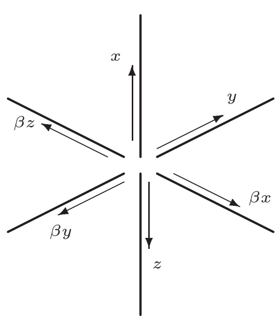
(b) $N_2^{(3)}$



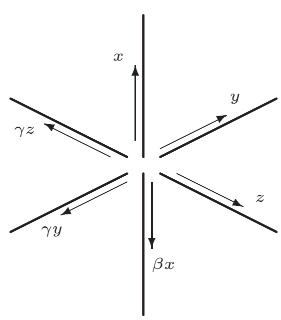
(c) $N_3^{(3)}$



(d) $N_4^{(3)}$



(e) $N_5^{(3)}$



(f) $N_6^{(3)}$

Fig.6.2 Barflies of three edges

Proof From Lemma 5.1, by basic transformation M can be transformed into a single vertex map. From Theorem 4.3, in view of

the nonorientability of M, the single vertex map is also nonorientable. Then from Lemma 5.2, by basic transformation the single vertex can be transformed into a single face map. From Theorem 4.3, this map is also nonorientable and hence a barfly N which satisfies(6.1). $\square$

This theorem enables us to restrict ourselves only to transform a barfly into another barfly under the basic equivalence.

VI.2 Simplified barflies

Let $Q_l = (\mathcal{X}_l, \mathcal{I}_l)$, $l \geq 1$, where

$$\mathcal{X}_l = \sum_{i=1}^{l} Kx_i, \tag{6.2}$$

and

$$\mathcal{I}_l = (\prod_{i=1}^{l} \langle x_i, \beta x_i \rangle), \tag{6.3}$$

they are called N-*standard map* . When $k = 1$, 2, 3 and 4, $Q_1 = N^{(1)}$, $Q_2 = N_2^{(2)}$, $Q_3 = N_1^{(3)}$ and Q_4 are, respectively, shown in (a), (b), (c) and (d) of Fig.6.3.

Lemma 6.2 For any $l \geq 1$, N-standard maps Q_l are all nonorientable.

Proof Because all Q_l, $l \geq 1$, are single vertex map and $\beta x_1 \in \{x_1\}_{\mathcal{I}_l}$, $l \geq 1$, from Lemma 6.1, they are all nonorientable. $\square$

Lemma 6.3 For any $l \geq 1$, N-standard maps Q_l are all with only one face.

Proof Because $Q_1 = N^{(1)}$, $Q_2 = N_2^{(2)}$ and $Q_3 = N_1^{(3)}$, from the two examples above, they are all with only one face. Their faces are $(x_1)_{\mathcal{I}_1\gamma} = (x_1, \alpha x_1)$, $(x_1)_{\mathcal{I}_2\gamma} = (\langle x_1 \rangle_{\mathcal{I}_1\gamma}, x_2, \alpha x_2) = (x_1, \alpha x_1, x_2, \alpha x_2)$ and $(x_1)_{\mathcal{I}_3\gamma} = (\langle x_1 \rangle_{\mathcal{I}_2\gamma}, x_3, \alpha x_3) = (x_1, \alpha x_1, x_2, \alpha x_2, x_3, \alpha x_3)$.

Assume, by induction, that

$$(x_1)_{\mathcal{I}_{l-1}\gamma} = (x_1, \alpha x_1, x_2, \alpha x_2, \cdots, x_{l-1}, \alpha x_{l-1}),$$

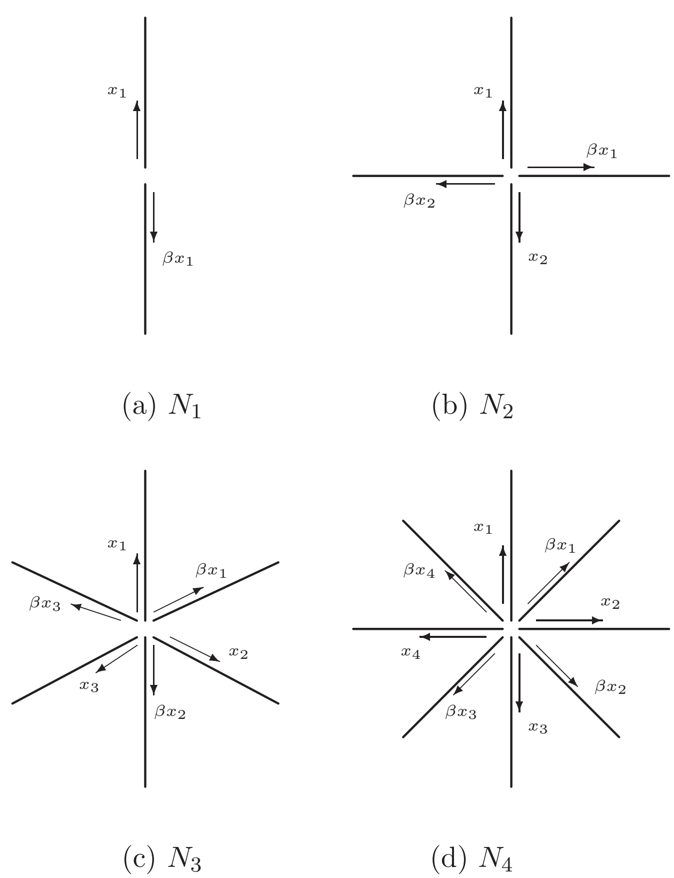
for $l \geq 4$. Since $\mathcal{I}_l = (\langle x_1 \rangle_{\mathcal{I}_{l-1}}, x_l, \beta x_l)$,

$$(x_1)_{\mathcal{I}_l\gamma} = (\langle x_1 \rangle_{\mathcal{I}_{l-1}\gamma}, (\mathcal{I}_l\gamma)\alpha x_{l-1}, \cdots).$$

And since $(\mathcal{I}_l\gamma)\alpha x_{l-1} = \mathcal{I}_l\beta x_{l-1} = x_l$, $(\mathcal{I}_l\gamma)x_l = \mathcal{I}_l\alpha(\beta x_l) = \alpha x_l$ and $(\mathcal{I}_l\gamma)\alpha x_l = \mathcal{I}_l\beta x_l = x_1$,

$$\begin{aligned}(x_1)_{\mathcal{I}_l\gamma} &= (\langle x_1 \rangle_{\mathcal{I}_{l-1}\gamma}, x_l, \alpha x_l)\\ &= (x_1, \alpha x_1, \cdots, x_{l-1}, \alpha x_{l-1}, x_l, \alpha x_l).\end{aligned}$$

Therefore, all N-standard maps are with only one face. $\square$



(a) N_1 (b) N_2

(c) N_3 (d) N_4

Fig.6.3 Simplified barfly

Because each N-standard map has only one face, from the two lemmas above, it is known that for any N-standard map is a barfly. And because each of such barflies has a simpler form, it is called a *simplified barfly*.

Since for any $l \geq 1$, the simplified barfly Q_l is with l edges, 1 vertex and 1 face, its Euler characteristic is

$$\chi(Q_l) = 2 - l. \tag{6.4}$$

Theorem 6.2 For any basic equivalent class of nonorientable maps, there exists at most one map which is a simplified barfly.

Proof By contradiction. Assume that there are two simplified barflies Q_i and Q_j, $i \neq j$, $i, j \geq 1$, in a basic equivalent class of nonorientable maps. From Theorem 4.5 and (6.4),

$$\chi(Q_i) = 2 - i = 2 - j = \chi(Q_j).$$

This implies $i = j$. A contradiction to the assumption. $\square$

In the following sections, it will be shown that there exists at least one map which is a simplified barfly in each basic equivalent class of nonorientable maps.

VI.3 Nonorientable rules

As mentioned above, this section is for establishing two basic rules of transforming a nonorientable single vertex map into another nonorientable single vertex map within basic equivalence.

Lemma 6.4 For a nonorientable single vertex map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{I})$, if $\mathcal{I} = (A, x, B, \beta x, C)$ where A, B and C are segments of linear order in the cycle $\mathcal{I}$ on $\mathcal{X}_{\alpha,\beta}$, then

$$\mathcal{I} \sim_{\mathrm{bc}} (A, \alpha B^{-1}, C, x, \beta x). \tag{6.5}$$

Note 6.1 For a segment $B = \langle \mathcal{I}x, \mathcal{I}^2 x, \cdots, \mathcal{I}^s x\rangle$, $\beta x = \mathcal{I}^{s+1} x$,

of linear order in the cycle of $\mathcal{I}$ on $\mathcal{I}_{\alpha,\beta}$, from Theorem 2.3,

$$\begin{aligned} &\langle \alpha\mathcal{I}^s x, \mathcal{I}(\alpha\mathcal{I}^s x), \cdots, \mathcal{I}^{s-1}(\alpha\mathcal{I}^s x)\rangle \\ &= \langle \alpha\mathcal{I}^s x, \alpha\mathcal{I}^{s-1}x, \cdots, \alpha\mathcal{I}x\rangle \\ &= \alpha B^{-1}. \end{aligned} \tag{6.6}$$

Proof Two steps expressed by claims are considered for transforming a nonorientable single vertex map into another nonorientable single vertex map under the basic equivalence.

Claim 1 $(A, x, B, \beta x, C) \sim_{\mathrm{bc}} (A, \alpha B^{-1}, \beta y, \alpha C^{-1}, y)$.

Proof By basic splitting an edge e_y between the two angles $\langle \alpha x, \mathcal{I}x\rangle$ and $\langle C, A\rangle$(*i.e.*, the angle between C and A) on $\mathcal{I} = (A, x, B, \beta x, C)$,

$$\begin{aligned} \mathcal{I} &\sim_{\mathrm{bc}} (A, x, y)(\gamma y, B, \beta x, C) \\ &= (A, x, y)(\beta y, \alpha C^{-1}, \gamma x, \alpha B^{-1}) \\ &= (y, A, x)(\gamma x, \alpha B^{-1}, \beta y, \alpha C^{-1}), \end{aligned}$$

as shown in (a) and (b) of Fig.6.4.

Because e_x is a link in $\mathcal{I}_1 = (y, A, x)(\gamma x, \alpha B^{-1}, \beta y, \alpha C^{-1})$, by basic contracting e_x,

$$\begin{aligned} \mathcal{I}_1 &\sim_{\mathrm{bc}} (y, A, \alpha B^{-1}, \beta y, \alpha C^{-1}) \\ &= (A, \alpha B^{-1}, \beta y, \alpha C^{-1}, y), \end{aligned}$$

as shown in (c) of Fig.6.4.

Claim 2 $\mathcal{I}_2 \sim_{\mathrm{bc}} (A, \alpha B^{-1}, C, \beta x, x)$ where

$$\mathcal{I}_2 = (A, \alpha B^{-1}, \beta y, \alpha C^{-1}, y).$$

Proof By basic splitting e_x between the two angles $(\gamma y, \mathcal{I}_2\beta y)$ and $(\alpha y, \mathcal{I}_2 y)$ on $(A, \alpha B^{-1}, \beta y, \alpha C^{-1}, y)$,

$$\begin{aligned} \mathcal{I}_2 &\sim_{\mathrm{bc}} (A, \alpha B^{-1}, \beta y, x)(\gamma x, \alpha C^{-1}, y) \\ &= (A, \alpha B^{-1}, \beta y, x)(\alpha y, C, \beta x) \\ &= (x, A, \alpha B^{-1}, \beta y)(\alpha y, C, \beta x), \end{aligned}$$

as shown in (d) and (e) of Fig.6.4.

Because e_y is a link in $\mathcal{I}_3 = (x, A, \alpha B^{-1}, \beta y)(\alpha y, C, \beta x)$, by basic contracting e_y,

$$\begin{aligned}\mathcal{I}_3 &\sim_{\text{bc}} (x, A, \alpha B^{-1}, C, \beta x)\\ &= (A, \alpha B^{-1}, C, \beta x, x),\end{aligned}$$

as shown in (f) of Fig.6.4.

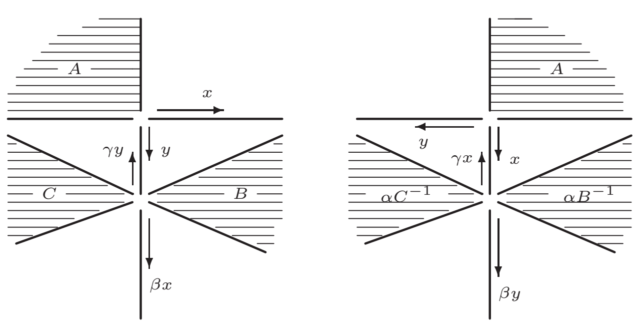
From Claim 1 and Claim 2,

$$\begin{aligned}\mathcal{I} &\sim_{\text{bc}} \mathcal{I}_1 \sim_{\text{bc}} \mathcal{I}_2 \sim_{\text{bc}} \mathcal{I}_3\\ &\sim_{\text{bc}} (A, \alpha B^{-1}, C, \beta x, x).\end{aligned}$$

This is (6.5). □

On the basis of the procedure in the proof of the lemma, the two claims show the following rules as basic equivalent transformations.

Nonorientable rule 1 On a nonorientable map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ unnecessary to have a single vertex, if $\beta x \in (x)_{\mathcal{P}}$, then the map M' obtained by translating x and βx in a direction via, respectively, segments C and D, and then by substituting αC^{-1} and αD^{-1} for, respectively, C(without βx) and D(without x) on $(x)_{\mathcal{P}}$ is basic equivalent to M, *i.e.*, $M' \sim_{\text{bc}} M$.



(a) (b)

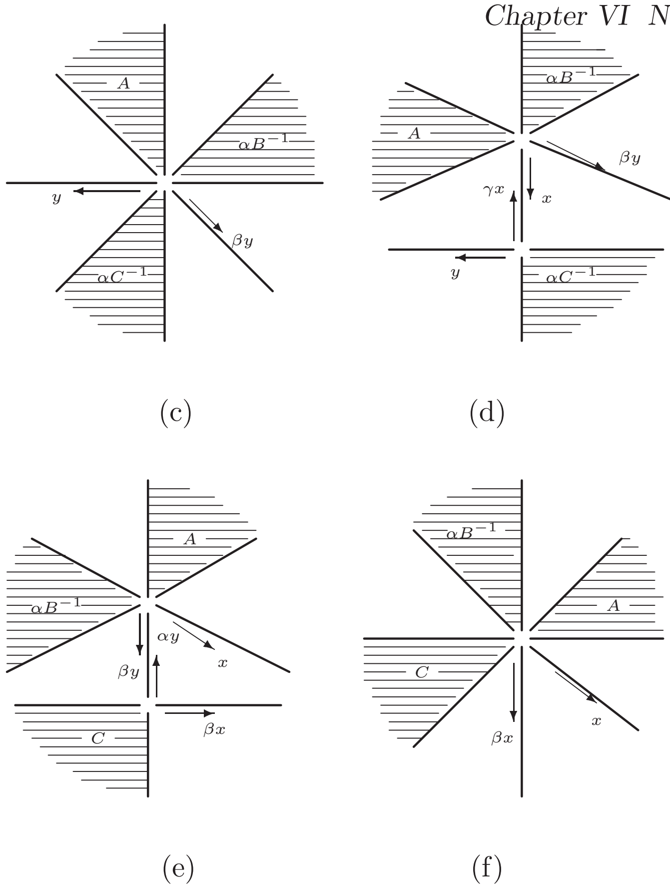


Fig.6.4 Claim 1 and Claim 2

This is, in fact, the Claim 1 above. However, from the proof of Claim 2 a much simpler rule can be extracted.

Nonorientable rule 2 On a nonorientable map $M = (\mathcal{X}, \mathcal{P})$ unnecessary to have a single vertex, if $\beta x \in (x)_{\mathcal{P}}$, then the map M' obtained by translating x(or βx) via a segment C, and then by substituting αC^{-1} for C without βx(or x) on $(x)_{\mathcal{P}}$ is basic equivalent to M, *i.e.*, $M' \sim_{\rm bc} M$.

It is seen that Nonorientable rule 1 can be done by employing Nonorientable rule 2 twice. Therefore, Nonorientable rule 2 is fundamental. From this point of view, the proof of Lemma 6.4 can be done only by Nonorientable rule 2.

Lemma 6.5 For a nonorientable single vertex map $(\mathcal{X}_{\alpha,\beta}, \mathcal{I})$,

$\gamma = \alpha\beta$, if

$$\mathcal{I} = (A, x, \beta x, y, z, \gamma y, \gamma z)$$

where A is a segment of linear order on $\mathcal{X}_{\alpha,\beta}$, then

$$\mathcal{I} \sim (A, x, \beta x, y, \beta y, z, \beta z). \tag{6.7}$$

Proof By basic splitting e_t between the two angles $(\alpha x, \beta x)$ and $(\alpha z, \gamma y)$ on $(A, x, \beta x, y, z, \gamma y, \gamma z)$,

$$\begin{aligned}\mathcal{I} &= (A, x, \beta x, y, z, \gamma y, \gamma z)\\ &\sim_{\mathrm{bc}} (\gamma y, \gamma z, A, x, t)(\gamma t, \beta x, y, z)\\ &= (\gamma y, \gamma z, A, x, t)(\gamma x, \beta t, \alpha z, \alpha y)\\ &= (t, \gamma y, \gamma z, A, x)(\gamma x, \beta t, \alpha z, \alpha y).\end{aligned}$$

Because e_x is a link in $\mathcal{I}_1 = (t, \gamma y, \gamma z, A, x)(\gamma x, \beta t, \alpha z, \alpha y)$, by contracting e_x,

$$\begin{aligned}\mathcal{I}_1 &\sim_{\mathrm{bc}} (t, \gamma y, \gamma z, A, \beta t, \alpha z, \alpha y)\\ &= (A, \beta t, \alpha z, \alpha y, t, \gamma y, \gamma z).\end{aligned}$$

By substituting $\langle A, \beta t\rangle$, $\langle \alpha y, t, \gamma y\rangle$ and $\langle \emptyset\rangle$ for , respectively, A, B and C in (6.5),

$$\mathcal{I}_1 \sim_{\mathrm{bc}} (A, \beta t, \beta y, \alpha t, y, z, \beta z).$$

Further, by substituting $\langle A, \beta t\rangle$, $\langle \alpha t\rangle$ and $\langle z, \beta z\rangle$ for, respectively, A, B and C in (6.5),

$$\begin{aligned}\mathcal{I}_1 &\sim_{\mathrm{bc}} (A, t, \beta t, y, \beta y, z, \beta z)\\ &= (A, x, \beta x, y, \beta y, z, \beta z).\end{aligned}$$

This is (6.7). $\square$

This lemma shows that in a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, $\gamma = \alpha\beta$, if

$$(x)_{\mathcal{P}} = (x, \beta x, y, z, \gamma y, \gamma z, A),$$

then the map obtained by substituting $\langle y, \beta y, z, \beta z\rangle$ for $\langle y, z, \gamma y, \gamma z\rangle$ on $(x)_{\mathcal{P}}$ is basic equivalent to M, *i.e.*, $M' \sim_{\mathrm{bc}} M$. This is usually called *nonorientable rule* 3.

Actually, nonorientable rule 3 can be also deduced from Nonorientable rule 2. Although Nonorientable rule 2 is fundamental, Nonorientable rule 1 and nonorientable rule 3 are more convenient for recursion.

VI.4 Nonorientable principles

In this section, barflies are only considered for this classification because it has been known that there is no loss of generality for general nonorientable maps.

Lemma 6.6 In a barfly $N = (\mathcal{X}_{\alpha,\beta}, \mathcal{I})$, there exists an element $x \in \mathcal{X}_{\alpha,\beta}$ such that

$$\mathcal{I} = (A, x, B, \beta x, C), \tag{6.8}$$

where A, B and C are segments of $\mathcal{I}$ on $\mathcal{X}_{\alpha,\beta}$.

Proof By contradiction. Since A, B and C are permitted to be empty, if no $x \in \mathcal{X}$ such that $\mathcal{I}$ satisfies (6.8), then from only one vertex, for any $x \in \mathcal{X}$, it is only possible that $\gamma x \in (x)_{\mathcal{I}}$ and $\beta x \notin (x)_{\mathcal{I}}$. Therefore, $(x)_{\Psi_{\{\mathcal{I},\gamma\}}}$ and $(\beta x)_{\Psi_{\{\mathcal{I},\gamma\}}}$ are the two orbits of $\Psi_{\{\mathcal{I},\gamma\}}$ on $\mathcal{X}_{\alpha,\beta}$. Thus, M is orientable. This is a contradiction to the nonorientability of a barfly. $\square$

Theorem 6.3 For any barfly $N = (\mathcal{X}_{\alpha,\beta}, \mathcal{I})$, there exists an integer $l \geq 1$ such that

$$\mathcal{I} \sim_{\mathrm{bc}} Q_l. \tag{6.9}$$

Proof From Lemma 6.6 and Lemma 6.4, it can assumed that

$$\mathcal{I} \sim_{\mathrm{bc}} (A, \prod_{j=1}^{i} \langle x_j, \beta x_j \rangle),$$

where i is as great as possible in this form. Naturally, $i \geq 1$.

From the maximality of i and only one vertex, $x \in A$ if, and only if, $\gamma x \in A$.

Two cases have to be discussed.

Case 1 If no element in A is interlaced, then from Corollary 5.2 and Corollary 5.1, (6.9) holds. Here, $l = i$.

Case 2 Otherwise, by Lemma 5.7(the reduced rule). it can be assumed that

$$\mathcal{I} \sim_{\mathrm{bc}} (B, \prod_{j=1}^{i} \langle x_j, \beta x_j \rangle \prod_{j=1}^{t} \langle y_j, z_j, \gamma y_j, \gamma z_j \rangle),$$

where t is as great as possible in this form. Naturally, $t \geq 1$. From the maximality of t, no element in B is interlaced. By Corollary 5.2 and Corollary 5.1,

$$\mathcal{I} \sim_{\mathrm{bc}} (\prod_{j=1}^{i} \langle x_j, \beta x_j \rangle \prod_{j=1}^{t} \langle y_j, z_j, \gamma y_j, \gamma z_j \rangle).$$

By nonorientable rule 3,

$$\mathcal{I} \sim_{\mathrm{bc}} (\prod_{j=1}^{2t+i} \langle x_j, \beta x_j \rangle) = \mathcal{I}_l.$$

From (6.2) and (6.3), this is (6.9) where $l = 2t + i$. □

On the basis of Theorem 6.1 and Theorem 6.3, it is know that there is at least one simplified barfly in each of basic equivalent classes for nonorientable maps.

VI.5 Nonorientable genus

Now, let us go back to general nonorientable maps for the invariants of determining the basic equivalent classes for nonorientable maps.

Theorem 6.4 For any nonorientable map $N = (\mathcal{X}, \mathcal{P})$ in a basic equivalent class, there has, and only has, an integer $l \geq 1$ such that the Euler characteristic

$$\chi(N) = 2 - l. \tag{6.10}$$

Proof From Theorem 6.3, there is a simplified barfly in a basic equivalent class of barflies. From Theorem 6.1, in each basic equivalent class of nonorientable maps, there has an integer $l \geq 1$ such that Q_l is in this class. On the other hand, from Theorem 6.2, only Q_l is in this class. Therefore, from (6.4) and Theorem 4.5, (6.10) is obtained. $\square$

This integer $l = 2 - \chi(N) \geq 1$ is called the *nonorientable genus* of the class N is in, or of N.

Now, it is seen from Chapters IV, V and VI that if the orientability of a map is defined to be 1, when the map is orientable; -1, when the map is nonorientable, then the *relative genus* of a map is the product of its orientability and its *absolute genus* (orientable genus, if the map is orientable; nonorientable genus, if the map is nonorientable). Thus, a basic equivalent class of maps(orientable and nonorientable) is determined by only its relative genus.

Activities on Chapter VI

V.6 Observations

O6.1 Think, is the map obtained by deleting an edge on a nonorientable map still nonorientable? If yes, explain the reason. Otherwise, provide an example.

O6.2 Think, the absolute genus of a map obtained via deleting an edge on a map(orientable and nonorientable) is at most 1 less than that of the original map. Explain the why.

O6.3 Think, is the map obtained by contracting an edge on a nonorientable map still nonorientable? If yes, explain the reason. Otherwise, provide an example.

O6.4 Think, the absolute genus of the map obtained via contracting an edge on a map(orientable and nonorientable) is at most 1 less than that of the original map. Explain the why.

O6.5 Observe if a nonorientable map can always be transformed into a barfly via only basic deleting and basic appending an edge. If it can, explain the reason. Otherwise, discuss what type of nonorientable maps can be done.

O6.6 Observe if a nonorientable map can always be transformed into a barfly via only basic contracting and basic splitting an edge. If it can, explain the reason. Otherwise, discuss what type of nonorientable maps can be done.

O6.7 Observe if there are other standard maps than simplified barflies for the classification of nonorientable maps under basic

equivalence.

O6.8 Consider how to derive the nonorientable rule 3 only from the nonorientable rule 2.

O6.9 For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, if the linear order

$$\langle A, x, B, \beta x, C\rangle \subseteq (x)_{\mathcal{P}}$$

is replaced by $\langle A, \alpha B^{-1}, C, x, \beta x\rangle$ to transform the permutation $\mathcal{P}$ on $\mathcal{X}_{\alpha,\beta}$ into permutation $\mathcal{P}'$ on $\mathcal{X}_{\alpha,\beta}$, then $M' = (\mathcal{X}, \mathcal{P}')$ is also a map. Is M' basic equivalent to M? If yes, explain the reason. Otherwise, provide an example.

O6.10 Observe that all the nonorientable rules 1–3 are valid for any nonorientable map not necessary to be of single vertex.

VI.7 Exercises

E6.1 By basic deleting and basic appending edge, prove that

$$(A, x, B, \alpha x, C) \sim_{\mathrm{bc}} (A, x, B, \beta C^{-1}, \alpha x).$$

E6.2 For a given nonorientable map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ and an element $x \in \mathcal{X}_{\alpha,\beta}$, the linear order $\langle x, y, \gamma, \gamma y\rangle \subseteq (x)_{\mathcal{P}}$ is replaced by $\langle x, \beta x, y, \beta y\rangle \subseteq (x)_{\mathcal{P}'}$ for obtaining $M' = (\mathcal{X}, \mathcal{P}')$. Prove that $M' \sim_{\mathrm{bc}} M$.

E6.3 By basic deleting and basic appending an edge, prove

$$(A, x, y, \gamma x, \gamma y, z, \alpha z) \sim_{\mathrm{bc}} (A, x, \alpha x, y, \alpha y, z, \alpha z).$$

E6.4 List all barflies of three edges rather than those in Example 2.

E6.5 Prove that for any two barflies, one can always be transformed into another only by the nonorientable rule 2.

The irreducibility appearing in what follows is in agreement with that in section V.7.

E6.6 Determine all irreducible barflies under the basic deleting and basic appending an edge.

E6.7 For a given integer $l \geq 1$, determine all the irreducible maps of absolute genus l under the basic deleting and basic appending an edge.

E6.8 Determine all irreducible barflies under the basic contracting and basic splitting an edge.

E6.9 For a given integer $l \geq 1$, determine all the irreducible maps of absolute genus l under the basic contracting and basic splitting an edge.

E6.10 Prove that for any nonorientable map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, there has, and only has, an integer $l \geq 1$ such that

$$\mathcal{P} \sim_{\mathrm{bc}} \begin{cases} (\prod_{i=1}^{s} \langle x_i, y_i, \beta x_i, \gamma y_i \rangle, x_{s+1}, \beta x_{s+1}), \\ \qquad \text{当 } l = 2s+1, s \geq 0; \\ (\prod_{i=1}^{s} \langle x_i, y_i, \beta x_i, \gamma y_i \rangle), \text{当 } l = 2s, s \geq 1. \end{cases} \tag{6.11}$$

Here, when $s = 0$,

$$\prod_{i=1}^{s} \langle x_i, y_i, \beta x_i, \gamma y_i \rangle = \emptyset.$$

E6.11 Prove that for any nonorientable map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, there has, and only has, an integer $l \geq 1$ such that

$$\begin{aligned} \mathcal{P} &\sim_{\mathrm{bc}} (x_1, x_2, \cdots, x_k, \beta x_k, \cdots, \beta x_2, \beta x_1) \\ &= (\prod_{i=1}^{k} x_i, \prod_{i=k}^{1} \beta x_i) \end{aligned}$$

by the nonorientable rules instead of Lemma 6.4.

E6.12 Prove that for any graph G, but a tree, G has its maximum nonorientable genus

$$l_{\mathrm{M}}(G) = \epsilon(G) - \nu(G) + 1 \tag{6.12}$$

where $\epsilon(G)$ and $\nu(G)$ are, respectively, the size(edge number) and the order(vertex number) of G.

VI.8 Researches

Similarly to Chapter V, among all nonorientable embeddings of a graph, the one with minimum (maximum) of absolute genus is called a *minimum* (*maximum*) *genus embedding*.

The genus of a minimum(maximum) genus embedding on nonorientable surfaces for a graph is called the *minimum* (*maximum*) *nonorientable genus* of the graph.

The minimum nonorientable genus of a graph is also called the *nonorientable genus* of the graph. If the minimum genus embedding is a nonorientable pan-tour(favorable) map, the the genus is called the *nonorientable pan-tour*(*favorable*) *genus*.

And the likes, *nonorientable pan-tour maximum genus*, *nonorientable tour genus* (or *nonorientable preproper genus*), *nonorientable tour maximum genus*, *etc*.

R6.1 Justify and recognize if a graph has a nonorientable embedding which is a pan-tour map.

R6.2 Justify and recognize if a graph has a nonorientable embedding which is a tour map.

R6.3 Determine the least upper bound and the greatest lower bound of the nonorientable pan-tour genus(or genera) for a graph(or a set of graphs).

R6.4 Determine the least upper bound and the greatest lower bound of the nonorientable tour genus(or genera) for a graph(or a set of graphs).

R6.5 Determine the least upper bound and the greatest lower bound of the nonorientable proper genus(or genera) for a graph(or a set of graphs).

Because it looks no much possibility to get a result simple as

shown in (6.12) for determining the nonorientable pan-tour maximum genus, nonorientable tour maximum genus and nonorientable proper maximum genus of a graph in general, only some types of graphs are available to be considered for such kind of result.

R6.6 Determine the least upper bound and the greatest lower bound of the nonorientable pan-tour maximum genus(or genera) for a graph(or a set of graphs).

R6.7 Determine the least upper bound and the greatest lower bound of the nonorientable tour maximum genus(or genera) for a graph(or a set of graphs).

R6.8 Determine the least upper bound and the greatest lower bound of the nonorientable proper maximum genus(or genera) for a graph(or a set of graphs).

R6.9 *Nonorientable pan-tour conjecture* (prove, or disprove). Any nonseparable graph has a nonorientable embedding which is a pan-tour map.

R6.10 *Nonorientable tour map conjecture* (prove, or disprove). Any nonseparable graph has a nonorientable embedding which is a tour map.

R6.11 *Nonorientable proper map conjecture* (prove,or disprove). Any nonseparable graph has a nonorientable embedding which is a proper map.

R6.12 *Nonorientable Small face proper map conjecture*(prove, or disprove). A nonseparable graph of order n has a nonorientable embedding which is a proper map with $n-1$ faces.

Chapter VII

Isomorphisms of Maps

- An isomorphism is defined for the classification of maps. A map is dealt with an isomorphic class of embeddings of the under graph of the map.
- Two maps are isomorphic if, and only if, their dual maps are isomorphic with the same isomorphism.
- Two types of efficient algorithms are designed for recognizing if two maps are isomorphic.
- Primal trail codes, or dual trail codes are used for justifying the isomorphism of two maps.
- Two pattern examples show how to to recognize and justify if two maps are isomorphic.

VII.1 Commutativity

In view of topology, the basic equivalent classes of maps are, in fact, a type of topological equivalent classes of 2-dimensional closed compact manifolds without boundary, or in brief surfaces.

Two embeddings of a graph explained in Chapter I are distinct if they are treated as 1-dimensional complexes to be non-equivalent under a topological equivalence.

If a map is dealt with an embedding of a graph on a surface, then two distinct maps are, of course, distinct embeddings of their under graph. However, the conversed case is not necessary to be true.

This Chapter is intended to introduce a type of combinatorial equivalence which is still seen as a type of topological equivalence but different from that for embeddings of a graph.

In general, the equivalence between two maps can be deduced from that between two embeddings of their under graph. However, the coversed case is not necessary to be true.

For two maps $M_1 = (\mathcal{X}_{\alpha,\beta}(X_1), \mathcal{P}_1)$ and $M_2 = (\mathcal{X}_{\alpha,\beta}(X_2), \mathcal{P}_2)$, if there exists a 1–to–1 correspondence (*i.e.*, bijection)

$$\tau \ : \ \mathcal{X}_{\alpha,\beta}(X_1) \ \longrightarrow \ \mathcal{X}_{\alpha,\beta}(X_2)$$

between $\mathcal{X}_{\alpha,\beta}(X_1)$ and $\mathcal{X}_{\alpha,\beta}(X_2)$ such that for any $x \in \mathcal{X}_{\alpha,\beta}(X_1)$,

$$\tau(\alpha x) = \alpha\tau(x), \ \tau(\beta x) = \beta\tau(x), \ \tau(\mathcal{P}_1 x) = \mathcal{P}_2\tau(x), \tag{7.1}$$

then τ is called an *isomorphism* from M_1 to M_2.

Lemma 7.1 If τ is an isomorphism from M_1 to M_2, then its inverse τ^{-1} exists, and τ^{-1} is an isomorphism from M_2 to M_1.

Proof Sinceτ is a bijection, τ^{-1} exists. And, τ^{-1} is also a 1–to–1 correspondence from M_2 to M_1. For any $y \in \mathcal{X}_{\alpha,\beta}(X_2)$, let $x = \tau^{-1}y \in \mathcal{X}_{\alpha,\beta}(X_1)$. Because $y = \tau x$ and τ is an isomorphism for M_1 to M_2, from (7.1),

$$\tau(\alpha x) = \alpha y, \ \tau(\beta x) = \beta y, \ \tau(\mathcal{P}_1 x) = \mathcal{P}_2 y.$$

Further, because τ^{-1} exists, then

$$\begin{aligned} \tau^{-1}(\alpha y) &= \alpha x = \alpha(\tau^{-1}y), \\ \tau^{-1}(\beta y) &= \beta x = \beta(\tau^{-1}y), \\ \tau^{-1}(\mathcal{P}_2 y) &= \mathcal{P}_1 x = \mathcal{P}_1(\tau^{-1}y). \end{aligned}$$

This implies τ^{-1} is an isomorphism from M_2 to M_1. $\square$

Based on this lemma, τ, or τ^{-1} can be called an isomorphism between M_1 and M_2.

Examle 7.1 Let $M_1 = (\mathcal{X}_1, \mathcal{P}_1)$ where

$$\mathcal{X}_1 = Kx_1 + Ky_1 + Kz_1 + Ku_1$$

and

$$\mathcal{P}_1 = (x_1, y_1, z_1)(u_1, \gamma z_1)(\beta u_1, \gamma x_1)(\gamma y_1),$$

and $M_2 = (\mathcal{X}_2, \mathcal{P}_2)$ where

$$\mathcal{X}_2 = Kx_2 + Ky_2 + Kz_1 + Ku_2$$

and

$$\mathcal{P}_2 = (y_2, z_2, x_2)(\gamma u_2, \beta x_2)(\alpha u_2, \beta z_2)(\beta y_2)$$

as shown in Fig.7.1.

First, let $\tau(x_1) = x_2$. from the first two relations in (7.1) and the property of Klein group, if τ is an isomorphism between M_1 and M_2, then

$$\begin{aligned}\tau(\alpha x_1) &= \alpha(\tau x_1) = \alpha x_2,\\ \tau(\beta x_1) &= \beta(\tau x_1) = \beta x_2,\\ \tau(\gamma x_1) &= \gamma(\tau x_1) = \gamma x_2,\end{aligned}$$

i.e., $\tau(Kx_1) = Kx_2$.

Then, from the third relation of (7.1),

$$\tau(y_1) = \tau(\mathcal{P}_1 x_1) = \mathcal{P}_2 \tau(x_1) = \mathcal{P}_2 x_2 = y_2.$$

Thus, $\tau(Ky_1) = Ky_2$. Similarly, from

$$\tau(z_1) = \tau(\mathcal{P}_1 y_1) = \mathcal{P}_2 \tau(y_1) = \mathcal{P}_2 y_2 = z_2,$$

$\tau(Kz_1) = Kz_2$, and from

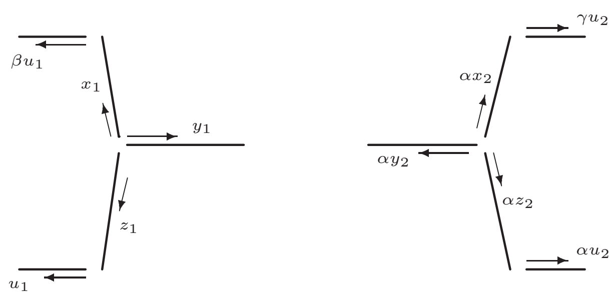
$$\tau(u_1) = \tau(\mathcal{P}_1 \gamma z_1) = \mathcal{P}_2 \tau(\gamma z_1) = \mathcal{P}_2 \gamma z_2 = u_2,$$

$\tau(Ku_1) = Ku_2$.

Finally, check that if the 1–to–1 correspondence τ from $\mathcal{X}_1$ to $\mathcal{X}_2$ satisfies $\tau\mathcal{P}_1 = \mathcal{P}_2$. In fact, from the conjugate axiom, it is only necessary to have

$$\begin{aligned}\tau\mathcal{P}_1 &= (\tau x_1, \tau y_1, \tau z_1)(\tau u_1, \tau\gamma z_1)(\tau\beta u_1, \tau\gamma x_1)(\tau\gamma y_1)\\ &= (x_2, y_2, z_2)(u_2, \gamma z_2)(\beta u_2, \gamma x_2)(\gamma y_2)\\ &= (y_2, z_2, x_2)(\gamma u_2, \beta x_2)(\alpha u_2, \beta z_2)(\beta y_2)\\ &= \mathcal{P}_2.\end{aligned}$$

Therefore, τ is an isomorphism between M_1 and M_1.



(a) M_1 (b) M_2

Fig.7.1 Two isomorphic maps

Note 7.1 If the two maps M_1 and M_2 in Example 1 are, respectively, seen as embeddings of their under graphs G_1 and G_2, then they are distinct. If Kx is represented by $x = (x^{+1}, x^{-1})$ where $x^{+1} = \{x, \alpha x\}$ and $x^{-1} = \{\beta x, \gamma x\}$, then the vertices μG_1 have their rotation as

$$(x_1^{+1}, y_1^{+1}, z_1^{+1}), (z_1^{+-1}, u_1^{+1}), (x_1^{-1}, u_1^{+-1}), (y_1^{-1}),$$

and hence μG_1 is on the projective plane (u_1, u_1). And the vertices of μG_2 have their rotation as

$$(x_2^{+1}, z_2^{+1}, y_2^{+1}), (z_1^{+-1}, u_1^{+1}), (x_1^{-1}, u_1^{+-1}), (y_1^{-1}),$$

and hence μG_2 is on the projective plane (u_1, u_1) as well.

However, the induced 1–to–1 correspondence $\tau|_\mu(\tau|_\mu(s_1) = s_2$, $s = x, y, z, u)$ from $\mu(G_1)$ to $\mu(G_2)$ has

$$\tau|_\mu(x_1^{+1}, y_1^{+1}, z_1^{+1}) = (x_2^{+1}, y_2^{+1}, z_2^{+1}) \neq (x_2^{+1}, z_2^{+1}, y_2^{+1}).$$

This implies that μG_1 and μG_2 are distinct.

Theorem 7.1 Let $M_1 = (\mathcal{X}_{\alpha,\beta}(X_1), \mathcal{P}_1)$ and $M_2 = (\mathcal{X}_{\alpha,\beta}(X_2), \mathcal{P}_2)$ be two maps. For a bijection $\tau : \mathcal{X}_{\alpha,\beta}(X_1) \longrightarrow \mathcal{X}_{\alpha,\beta}(X_2)$, τ is an iso-

morphism if, and only if, the diagrams

$$\begin{array}{ccc} \mathcal{X}_{\alpha,\beta}(X_1) & \xrightarrow{\quad\tau\quad} & \mathcal{X}_{\alpha,\beta}(X_2) \\ {\scriptstyle\eta_1}\downarrow & & {\scriptstyle\eta_2}\downarrow \\ \mathcal{X}_{\alpha,\beta}(X_1) & \xrightarrow{\quad\tau\quad} & \mathcal{X}_{\alpha,\beta}(X_2) \end{array} \tag{7.2}$$

for $\eta_1 = \eta_2 = \alpha$, $\eta_1 = \eta_2 = \beta$, and for $\eta_1 = \mathcal{P}_1$ and $\eta_2 = \mathcal{P}_2$, are all *commutative* , *i.e.*, all paths with the same initial object and the same terminal object have the same effect.

Proof Necessity. From the first relation in (7.1), for any $x \in \mathcal{X}_{\alpha,\beta}(X_1)$, $\tau(\alpha x) = \alpha(\tau x)$. That is to say the result of composing the mappings on the direct path

$$\mathcal{X}_{\alpha,\beta}(X_1) \xrightarrow{\alpha} \mathcal{X}_{\alpha,\beta}(X_1) \xrightarrow{\tau} \mathcal{X}_{\alpha,\beta}(X_2)$$

is the same as the result of composing the mappings on the direct path

$$\mathcal{X}_{\alpha,\beta}(X_1) \xrightarrow{\tau} \mathcal{X}_{\alpha,\beta}(X_2) \xrightarrow{\alpha} \mathcal{X}_{\alpha,\beta}(X_2).$$

Therefore, (7.2) is commutative for $\eta_1 = \eta_2 = \alpha$.

Similarly, from the second and the third relations in (7.1), the commutativity for $\eta_1 = \eta_2 = \beta$, and for $\eta_1 = \mathcal{P}_1$ and $\eta_2 = \mathcal{P}_2$ are obtained.

Sufficiency. On the basis of (7.2), the three relations in (7.1) can be induced from the commutativity for $\eta_1 = \eta_2 = \alpha$, for $\eta_1 = \eta_2 = \beta$, and for $\eta_1 = \mathcal{P}_1$ and $\eta_2 = \mathcal{P}_2$. This is the sufficiency $\square$

VII.2 Isomorphism theorem

Because the isomorphism between two maps determines an equivalent relation, what has to be considered for the equivalence is the equivalent classes, called *isomorphic classes* of maps. Two maps are said to be different if they are in different isomorphic classes. In order to clarify the isomorphic classes of maps, invariants should be investigated. In this and the next sections, a sequence of elements with

its length half the cardinality of the ground set. In fact, this implies that the isomorphic class can be determined by a polynomial of degree as a linear function of half the cardinality of the ground set for both orientable and nonorientable maps.

Lemma 7.2 If two maps M_1 and M_2 are isomorphic, then M_1 is orientable if, and only if, M_2 is orientable.

Proof Let $M_i = (\mathcal{X}_i, \mathcal{P}_i)$, $i = 1, 2$. Assume τ is an isomorphism from M_1 to M_2. From (7.2), $\tau\alpha = \alpha\tau$,$\tau\beta = \beta\tau$ and $\tau\mathcal{P}_1 = \mathcal{P}_2\tau$, *i.e.*, $\tau\alpha\tau^{-1} = \alpha$, $\tau\beta\tau^{-1} = \beta$ and $\tau\mathcal{P}_1\tau^{-1} = \mathcal{P}_2$.

Necessity. Since M_1 is orientable, from Theorem 4.1, permutation group $\Psi_1 = \Psi_{\{\mathcal{P}_1,\gamma\}}$ has two orbits $(x_1)_{\Psi_1}$ and $(\alpha x_1)_{\Psi_1}$, $x_1 \in \mathcal{X}_1$ on $\mathcal{X}_1$. And, since $\tau\alpha\tau^{-1} = \alpha$ and $\tau\beta\tau^{-1} = \beta$,

$$\begin{aligned}\tau\gamma\tau^{-1} &= \tau(\alpha\beta)\tau^{-1} = \tau(\alpha\tau^{-1}\tau\beta)\tau^{-1}\\ &= (\tau\alpha\tau^{-1})(\tau\beta\tau^{-1}) = \alpha\beta\\ &= \gamma.\end{aligned}$$

By considering $\tau\mathcal{P}_1\tau^{-1} = \mathcal{P}_2$, for any $\psi_1 \in \Psi_1$,

$$\tau\psi_1\tau^{-1} = \psi_2 \in \Psi_2.$$

Therefore, Ψ_2 also has two orbits on $\mathcal{X}_2$, *i.e.*, $(x_2)_{\Psi_2}$ and $(\alpha x_2)_{\Psi_2}$, where $x_2 = \tau x_1 \in \mathcal{X}_2$. This implies that M_2 is orientable as well.

Sufficiency. Because of the symmetry of τ between M_1 and M_2, the sufficiency is deduced from the necessity. $\square$

For a map $M = (\mathcal{X}, \mathcal{P})$ where $\nu(M)$, $\epsilon(M)$ and $\phi(M)$ stand for, respectively, the *order*(vertex number), the *size*(edge number) and the *coorder*(face number) of M.

Lemma 7.3 If two maps M_1 and M_2 are isomorphic, then

$$\nu(M_1) = \nu(M_2),\ \epsilon(M_1) = \epsilon(M_2),\ \phi(M_1) = \phi(M_2). \tag{7.3}$$

Proof Let $M_i = (\mathcal{X}_i, \mathcal{P}_i)$, $i = 1, 2$. Assume τ is an isomorphism from M_1 to M_2. From the commutativity for $\eta_1 = \mathcal{P}_1$ and $\eta_2 = \mathcal{P}_2$ in

(7.2), $\tau\mathcal{P}_1\tau^{-1} = \mathcal{P}_2$. Then, for any integer $n \geq 1$ by induction,

$$\begin{aligned}\tau(\mathcal{P}_1)^n\tau^{-1} &= \tau((\mathcal{P}_1)^{n-1}\mathcal{P}_1)\tau^{-1}\\ &= \tau((\mathcal{P}_1)^{n-1}\tau^{-1}\tau\mathcal{P}_1)\tau^{-1}\\ &= (\tau(\mathcal{P}_1)^{n-1}\tau^{-1})(\tau\mathcal{P}_1)\tau^{-1})\\ &= (\tau(\mathcal{P}_1)^{n-1}\tau^{-1})(\tau\mathcal{P}_1)\tau^{-1})\\ &= (\mathcal{P}_2)^{n-1}\mathcal{P}_2\\ &= \mathcal{P}_2^n.\end{aligned}$$

Therefore, for any $x_1 \in \mathcal{X}_1$, $\tau x_1 = x_2$,

$$\tau(x_1)_{\mathcal{P}_1} = (\tau x_1)_{\tau\mathcal{P}_1\tau^{-1}} = (x_2)_{\mathcal{P}_2}.$$

Because a 1–to–1 correspondence on vertices between M_1 and M_2 is induced from this, $\nu(M) = \nu(M)$.

Similarly, from $\tau\gamma\tau^{-1} = \gamma$ and $\tau(\mathcal{P}_1)^n\tau^{-1} = (\mathcal{P}_2)^n$,

$$\tau(\mathcal{P}_1\gamma)\tau^{-1} = \mathcal{P}_2\gamma.$$

Further, for any integer $n \geq 1$, $\tau(\mathcal{P}_1\gamma)^n\tau^{-1} = (\mathcal{P}_2\gamma)^n$. This provides

$$\tau(x_1)_{\mathcal{P}_1\gamma} = (\tau x_1)_{\tau(\mathcal{P}_1\gamma)\tau^{-1}} = (x_2)_{\mathcal{P}_2\gamma}$$

as a 1–to–1 correspondence on faces between M_1 and M_2. Therefore, $\phi(M_1) = \phi(M_2)$.

Finally, from $\tau\alpha\tau^{-1} = \alpha$ and $\tau\beta\tau^{-1} = \beta$ and hence $\tau\gamma\tau^{-1} = \gamma$, for any $x_1 \in \mathcal{X}_1$, $x_2 = \tau x_1$ implies $\tau K x_1 = K x_2$. This provides a 1–to–1 correspondence on edges between M_1 and M_2. therefore, $\epsilon(M_1) = \epsilon(M_2)$. □

For a map $M = (\mathcal{X}, \mathcal{P})$, the Euler characteristic given by (4.1) is $\chi(M) = \nu(M) - \epsilon(M) + \phi(M)$ where $\nu(M)$, $\epsilon(M)$ and $\phi(M)$ are, respectively, the order, the size and the co-order of M.

Corollary 7.1 If two maps M_1 and M_2 are isomorphic, then

$$\chi(M_1) = \chi(M_2). \tag{7.4}$$

Proof A direct result of Lemma 7.3. $\square$

For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, let $M^* = (\mathcal{X}^*_{\alpha,\beta}, \mathcal{P}^*)$ be the dual of M. It is, from Chapter III, known that $M^* = (\mathcal{X}_{\beta,\alpha}, \mathcal{P}\gamma)$.

Theorem 7.2 Maps M_1 and M_2 are isomorphic if, and only if, their duals M_1^* and M_2^* are isomorphic.

Proof Let $M_i = (\mathcal{X}^{(i)}_{\alpha,\beta}, \mathcal{P}_i)$, $i = 1, 2$, then $M_i^* = ({\mathcal{X}^{(i)}_{\alpha,\beta}}^*, \mathcal{P}_i^*)$, $i = 1, 2$, where ${\mathcal{X}^{(i)}_{\alpha,\beta}}^* = \mathcal{X}^{(i)}_{\beta,\alpha}$ and $\mathcal{P}_i^* = \mathcal{P}_i\gamma$, $i = 1, 2$.

Necessity. Suppose τ is an isomorphic between M_1 and M_2, then from Theorem 7.1,

$$\tau\alpha\tau^{-1} = \alpha,\ \tau\beta\tau^{-1} = \beta,\ \tau\mathcal{P}_1\tau^{-1} = \mathcal{P}_2.$$

On the basis of this, for any $x_1 \in {\mathcal{X}^{(1)}_{\alpha,\beta}}^* = \mathcal{X}^{(1)}_{\beta,\alpha}$ and $x_2 = \tau x_1 \in {\mathcal{X}^{(2)}_{\alpha,\beta}}^* = \mathcal{X}^{(2)}_{\beta,\alpha}$,

$$\begin{aligned}\tau K^* x_1 &= \tau\{x_1, \beta x_1, \alpha x_1, \gamma x_1\} = \{\tau x_1, \tau\beta x_1, \tau\alpha x_1, \tau\gamma x_1\}\\ &= \{x_2, \beta x_2, \alpha x_2, \gamma x_2\} = K^* x_2,\end{aligned}$$

and

$$\begin{aligned}\tau\mathcal{P}_1^*\tau^{-1} &= \tau(\mathcal{P}_1\gamma)\tau^{-1} = \tau(\mathcal{P}_1\tau^{-1}\tau\gamma)\tau^{-1}\\ &= (\tau\mathcal{P}_1\tau^{-1})(\tau\gamma\tau^{-1}) = \mathcal{P}_2\gamma\\ &= \mathcal{P}_2^*.\end{aligned}$$

This implies that the diagram

$$\begin{tikzcd} {\mathcal{X}^{(1)}_{\alpha,\beta}}^* \arrow[r, "\tau"] \arrow[d, "\eta_1"'] & {\mathcal{X}^{(2)}_{\alpha,\beta}}^* \arrow[d, "\eta_2"] \\ {\mathcal{X}^{(1)}_{\alpha,\beta}}^* \arrow[r, "\tau"] & {\mathcal{X}^{(2)}_{\alpha,\beta}}^* \end{tikzcd} \tag{7.5}$$

are all commutative for $\eta_1 = \eta_2 = \beta$, for $\eta_1 = \eta_2 = \alpha$, and for $\eta_1 = \mathcal{P}_1^*$ and $\eta_2 = \mathcal{P}_2^*$. therefore, from Theorem 7.1, τ is an isomorphism between M_1^* and M_2^* in its own right.

Sufficiency. from the symmetry of duality, the sufficiency is deduced form the necessity. $\square$

Let $M_i = (\mathcal{X}^{(i)}_{\alpha,\beta}, \mathcal{P}_i)$, and $M^*_i = ({\mathcal{X}^{(i)}_{\alpha,\beta}}^*, \mathcal{P}^*_i)$ where ${\mathcal{X}^{(i)}_{\alpha,\beta}}^* = \mathcal{X}^{(i)}_{\beta,\alpha}$ and $\mathcal{P}^*_i = \mathcal{P}_i\gamma$, $i = 1, 2$.

Corollary 7.2 A bijection $\tau : \mathcal{X}^{(1)}_{\alpha,\beta} \longrightarrow \mathcal{X}^{(2)}_{\alpha,\beta}$ is an isomorphism between maps M_1 and M_2 if, and only if, τ is an isomorphism between maps M^*_1 and M^*_2.

Proof A direct result in the proof of Theorem 7.2. □

VII.3 Recognition

Although some invariants are provided, they are still far from determining an isomorphism between two maps in the last section.

In fact, it will be shown in this section that an isomorphism between two maps can be determined by the number of invariants dependent on their size, *i.e.*, a sequence of invariants in a number as a function of their size.

In order to do this, algorithms are established for justifying and recognizing if two maps are isomorphic. In other words, an isomorphism can be found between two maps if any; or no isomorphism exits at all otherwise.

Generally speaking, since the ground set of a map is finite, *i.e.*, its cardinality is 4ϵ, ϵ is the size of the map, in a theoretical point of view, there exists a permutation which corresponds to an isomorphism among all the $(4\epsilon)!$ permutations if any, or no isomorphism at all between two maps otherwise. However, this is a impractical way even on a modern computer.

Our purpose is to establish an algorithm directly with the amount of computation as small as possible without counting all the permutations.

Here, two types of algorithms are presented. One is called *vertex-algorithm* based on (7.2). Another is called *face-algorithm* based on (7.5).

Their clue is as follows. For two maps $M_1 = (\mathcal{X}_1, \mathcal{P}_1)$ and $M_2 =$

$(\mathcal{X}_2, \mathcal{P}_2)$, from Lemma 7.3, only necessary to consider $|\mathcal{X}_1| = |\mathcal{X}_2|$ because the cardinality is an invariant under an isomorphism.

First, choose $x_1 \in \mathcal{X}_1$ and $y_1 \in \mathcal{X}_2$ (a trick should be noticed here!).

Then, start, respectively, from x_1 and y_1 on M_1 and M_2 by a certain rule (algorithms are distinguished by rules). Arrange the orbits $\{x_1\}_{\Psi_{\{\mathcal{P}_1,\gamma\}}}$ and $\{y_1\}_{\Psi_{\{\mathcal{P}_2,\gamma\}}}$ as cycles. If

$$\tau(x_1)_{\Psi_{\{\mathcal{P}_1,\gamma\}}} = (y_1)_{\Psi_{\{\mathcal{P}_2,\gamma\}}}, \tag{7.6}$$

can be induced from $y_1 = \tau(x_1)$, then stop. Otherwise, choose another y_1(a trick!). Go no the procedure on M_2 until every possible y_1 has be chosen.

Finally, if stops at the latter, then it is shown that M_1 and M_2 are not isomorphic, and denoted by $M_1 \neq M_2$; otherwise, an isomorphism between M_1 and M_2 is done from (7.6), denoted by $M_1 = M_2$.

Algorithm 7.1 Based on vertices, determine if two maps are isomorphic.

Given two maps $P = (\mathcal{X}, \mathcal{P})$ and $Q = (\mathcal{Y}, \mathcal{Q})$, and their order, size and co-order are all equal(otherwise, not isomorphic!). In convenience, for any $x \in \mathcal{X}$, let $|x| = |\{x\}_{\mathcal{P}}|$, *i.e.* , the valency of vertex $(x)_{\mathcal{P}}$.

Initiation Given $x \in \mathcal{X}$, choose $y \in \mathcal{Y}$. Let $\tau(x) = y$ and $\tau Kx = Ky$. Label both x and y by 1. Naturally, $Kx = Ky = K1 = \{1, \alpha 1, \beta 1, \gamma 1\}$ (Here, the number 1 deals with a symbol!). Label $(x)_{\mathcal{P}}$ by 0, then $x = 1$ is the first element coming to vertex 0. By (v, t_v) denote that t_v is the first element coming to vertex v.

Let S be a sequence of symbols storing numbers and symbols and l, the maximum of labels on all the edges with a label. Here, $S = \emptyset$, $l = 1$ and the minimum of labels among all labelled but not passed vertices $n = 0$. If vertex $(\gamma 1)_{\mathcal{P}} = (1)_{\mathcal{P}}$, the maximum vertex label $m = 0$; otherwise, label vertex $(\gamma 1)_{\mathcal{P}}$ by 1, $m = 1$.

Proceeding When all vertices are labelled as used, then go to Halt (1).

For n, let s_P and s_Q be, respectively, the number of edges without label on $(\gamma t_n)_{\mathcal{P}}$ and $(\gamma t_n)_{\mathcal{Q}}$.

If $s_P \neq s_Q$, when no y can be chosen, then goto Halt (2); otherwise, choose another y and then goto Initiation.

In the direction starting from γt_n, label those edges by $l+1, \cdots, l+s$,$s = s_P = s_Q \geq 0$ in order. Thus, two linear orders of elements with numbers labelled

$$\langle \gamma t_n, \mathcal{P}\gamma t_n, \cdots, \mathcal{P}^{-1}\gamma t_n \rangle$$

and

$$\langle \gamma t_n, \mathcal{Q}\gamma t_n, \cdots, \mathcal{Q}^{-1}\gamma t_n \rangle$$

are obtained.

If the two are not equal, when no y is available to choose, then goto Halt (2); otherwise, choose another y and then goto Initiation.

Put this linear order into S as last part and then substitute the extended sequence for S. In the meantime, label $K(l+1)$, $K(l+2)$, $\cdots$, $K(l+s)$ on P and Q. Substitute $l+s$ for l. Mark vertex n as used. Substitute $n+1$ for n. Let r be the number of vertices without label in

$$(\gamma(l+1))_{\mathcal{P}}, \cdots, (\gamma(l+s))_{\mathcal{P}},$$

and label them as $m+1$,$\cdots$,$m+r$ in order. Substitute $m+r$ for m. Go on the Proceeding.

Halt (1) Output S. (2) P and Q are not isomorphic.

About Algorithm 7.1, from the way of choosing y, each element in the ground set is passed through at most once. So there exists a constant c such that the amount of computation is at most $c|\mathcal{X}|$. Since the worst case is for y chooses all over the ground set $\mathcal{Y}$,the total amount of computation is at most $c|\mathcal{X}|^2$. Because of $|\mathcal{X}| = 4\epsilon$ where ϵ is the size of the map, this amount is with its order as the size squared, *i.e.*, $O(\epsilon^2)$.

As described above, if checking all possibilities of $|\mathcal{Y}|!$, by Stirling

formula,

$$|\mathcal{Y}|! \sim \sqrt{2\pi}e^{-|\mathcal{Y}|}|\mathcal{Y}|^{|\mathcal{Y}|-\frac{1}{2}}$$
$$>> O(e^{|\mathcal{Y}|}) >> O(\epsilon^{\epsilon})$$
$$>> O(\epsilon^2)$$

when $|\mathcal{Y}| = |\mathcal{X}| = 4\epsilon$ is large enough. Thus, this algorithm is much efficient.

Algorithm 7.2 Based on faces, determine if two maps are isomorphic.

Given two maps $P = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ and $Q = (\mathcal{Y}_{\alpha,\beta}, \mathcal{Q})$, and their order, size and co-order are all equal(otherwise, not isomorphic!). For convenience, let $\mathcal{X} = \mathcal{X}_{\alpha,\beta}$, $\mathcal{Y} = \mathcal{Y}_{\alpha,\beta}$ and for any $x \in \mathcal{X}$, let $|x| = |\{x\}_{\mathcal{P}\gamma}|$, *i.e.*, the valency of face $(x)_{\mathcal{P}\gamma}$ where $\gamma = \alpha\beta$.

Initiation Given $x \in \mathcal{X}$, choose $y \in \mathcal{Y}$. Let $\tau(x) = y$ and $\tau Kx = Ky$. Label both x and y by 1. Naturally, $Kx = Ky = K1 = \{1, \alpha 1, \beta 1, \gamma 1\}$ (Here, the number 1 deals with a symbol!). Label $(x)_{\mathcal{P}|ga}$ by 0, then $x = 1$ is the first element coming to face 0. By (f, t_f) denote that t_f is the first element coming to face f.

Let T be a sequence of symbols storing numbers and symbols and l, the maximum of labels over all the edges with a label. Here, $T = \emptyset$, $l = 1$ and the minimum of labels among all labelled but not passed faces $n = 0$. If face $(\gamma 1)_{\mathcal{P}\gamma} = (1)_{\mathcal{P}\gamma}$, the maximum face label $m = 0$; otherwise, label face $(\gamma 1)_{\mathcal{P}\gamma}$ by 1, $m = 1$.

Proceeding When all faces are labelled as used, then go to Halt (1).

For n, let s_P and s_Q be, respectively, the number of edges without label on $(\gamma t_n)_{\mathcal{P}\gamma}$ and $(\gamma t_n)_{\mathcal{Q}\gamma}$.

If $s_P \neq s_Q$, when no y can be chosen, then goto Halt (2); otherwise, choose another y and then goto Initiation.

In the direction starting from γt_n, label those edges by $l+1, \cdots, l+s$,$s = s_P = s_Q \geq 0$ in order. Thus, two linear orders of elements with numbers labelled

$$\langle \gamma t_n, \mathcal{P}\gamma\gamma t_n, \cdots, \mathcal{P}\gamma^{-1}\gamma t_n \rangle$$

and

$$\langle \gamma t_n, \mathcal{Q}\gamma\gamma t_n, \cdots, \mathcal{Q}\gamma^{-1}\gamma t_n \rangle$$

are obtained.

If the two are not equal, when no y is available to choose, then goto Halt (2); otherwise, choose another y and then goto Initiation.

Put this linear order into S as last part and then substitute the extended sequence for S. In the meantime, label $K(l+1)$, $K(l+1)$, $\cdots$, $K(l+s)$ on P and Q. Substitute $l+s$ for l. Mark n as used. Substitute $n+1$ for n. Let r be the number of vertices without label in

$$(\gamma(l+1))_{\mathcal{P}}, \cdots, (\gamma(l+s))_{\mathcal{P}},$$

and label them as $m+1,\cdots,m+r$ in order. Substitute $m+r$ for m. Go on the Proceeding.

Put this linear order into T as last part and then substitute the extended sequence for T. In the meantime, label $K(l+1)$, $K(l+2)$, $\cdots$, $K(l+s)$ on P and Q. Substitute $l+s$ for l. Mark face n as used. Substitute $n+1$ for n. Let r be the number of faces without label in

$$(\gamma(l+1))_{\mathcal{P}}, \cdots, (\gamma(l+s))_{\mathcal{P}},$$

and label them as $m+1,\cdots,m+r$ in order. Substitute $m+r$ for m. Go on the Proceeding.

Halt (1) Output T. (2) P and Q are not isomorphic.

About Algorithm 7.2, it can be seen as the dual of Algorithm 7.1. The amount of its computation is also estimated as $O(\epsilon^2)$.

Note 7.2 This two algorithms suggest us that whenever a cyclic order of edges at each vertex is given, an efficient algorithm for justifying and recognizing if two graphs are isomorphic within the cyclic order at each vertex can be established. By saying an algorithm *efficient*, it is meant that there exists an constant c such that the amount of its computation is about $O(\epsilon^c)$, ϵ is the size of the graphs.

If without considering the limitation of a cyclic order at each vertex, no efficient algorithm for an isomorphism of two graphs has been

found yet up to now. However, a new approach is, from what has been discussed here, provided for further investigation of an isomorphism between two graphs.

VII.4 Justification

In this section, it is shown that the two algorithms described in the last section can be used for justifying and recognizing whether, or not, two maps are isomorphic.

Lemma 7.4 Let S and T are, respectively, the outputs of Algorithm 7.1 and Algorithm 7.2 at Halt (1), then

(i) Elements in S and T are all in the same orbit of group $\Psi_{\{\mathcal{P},\gamma\}}$ on $\mathcal{X}$.

(ii) S forms an orbit of group $\Psi_{\{\mathcal{P},\gamma\}}$ on $\mathcal{X}$ if, and only if, T forms an orbit of group $\Psi_{\{\mathcal{P},\gamma\}}$ on $\mathcal{X}$;

(iii) S forms an orbit of group $\Psi_{\{\mathcal{P},\gamma\}}$ on $\mathcal{X}$ if, and only if, for any $x \in S$, $\gamma x \in S$.

Proof (i) From the proceedings of the two algorithms, it is seen that from an element only passes through γ and $\mathcal{P}$(Algorithm 7.1), or γ and $\mathcal{P}\gamma$(Algorithm 7.2) for getting an element in S, or T. Because $\gamma, \mathcal{P}, \mathcal{P}\gamma \in \Psi_{\{\mathcal{P},\gamma\}}$ and $\gamma^2 = 1$, elements in S and T are all in the same orbit of group $\Psi_{\{\mathcal{P},\gamma\}}$ on $\mathcal{X}$.

(ii) Necessity. Because S forms an orbit of group $\Psi_{\{\mathcal{P},\gamma\}}$ on $\mathcal{X}$, and from Algorithm 7.1, S contains half the elements of $\mathcal{X}$, by Lemma 4.1, group $\Psi_{\{\mathcal{P},\gamma\}}$ has two orbits on $\mathcal{X}$. This implies in the orientable case. Thus, from (i), T forms an orbit of group $\Psi_{\{\mathcal{P},\gamma\}}$ on $\mathcal{X}$ as well.

Sufficiency. On the basis of duality, it is deduced from the necessity.

(iii) Necessity. Since S forms an orbit of group $\Psi_{\{\mathcal{P},\gamma\}}$ on $\mathcal{X}$ and S contains only half the elements of $\mathcal{X}$, by Lemma 4.1, group $\Psi_{\{\mathcal{P},\gamma\}}$ has two orbits on $\mathcal{X}$. From the orientability, for any $x \in S$, $\gamma x \in S$.

Sufficiency. Since for any $x \in S$, $\gamma x \in S$, and S only contains

half the elements of $\mathcal{X}$, by Corollary 4.1, it is only possible that S itself forms an orbit of group $\Psi_{\{\mathcal{P},\gamma\}}$ on $\mathcal{X}$. □

For nonorientable maps, such two algorithms have their outputs S and T also containing half the elements of $\mathcal{X}$ but not forming an orbit of group $\Psi_{\{\mathcal{P},\gamma\}}$.

Lemma 7.5 Let S and T are, respectively, the outputs of Algorithm 7.1 and Algorithm 7.2 at Halt (1). And , let G_S and G_T be, respectively, the graphs induced by elements in S and T, then $G_S = G_T = G(P)$.

Proof From Lemma 7.4(i), by the procedures of the two algorithms, because the intersection of each of S and T with any quadricell consists of two elements incident the two ends of the edge, S, T as well, is incident to all edges with two ends of each edge in map P.

Therefore, $G_S = G_T = G(P)$. □

Theorem 7.3 The output S of Algorithm 7.1 at Halt (1) induces an isomorphism between maps P and Q. Halt (2) shows that maps P and Q are not isomorphic.

Proof Let τ be a mapping from $\mathcal{X}$ to $\mathcal{Y}$ such that the image and the co-image are with the same label. From the transitivity of a map, τ is a bijection. Because $\tau Kx = K\tau x$, $x \in \mathcal{X}$, then $\tau\alpha\tau^{-1} = \alpha$ and $\tau\beta\tau^{-1} = \beta$. And in the Proceeding, for labelling a vertex $(x)_{\mathcal{P}}$, $\tau(x)_{\mathcal{P}} = (\tau x)_{\mathcal{Q}}$. From Lemma 7.5, this implies that $\tau\mathcal{P}\tau^{-1} = \mathcal{Q}$. Based on Theorem 7.1, τ is an isomorphism between P and Q. This is the first statement.

By contradiction to prove the second statement. Assume that there is an isomorphism τ between P and Q. If $\tau(x) = y$, then by Algorithm 7.1 the procedure should terminate at Halt (1). However, a termination at Halt (2) shows that for any $x \in \mathcal{X}$, there is no elements in $\mathcal{Y}$ corresponding to x in an isomorphism between maps P and Q, and hence it is impossible to terminate at Halt (1). This is a contradiction.

Therefore, the theorem is true. $\square$

Although the theorem below has its proof with a similar reasoning, in order to understand the precise differences the proof is still in a detailed explanation.

Theorem 7.4 The output T of Algorithm 7.2 at Halt (1) induces an isomorphism between maps P and Q. Halt (2) shows that maps P and Q are not isomorphic.

Proof Let τ be a mapping from $\mathcal{X}$ to $\mathcal{Y}$ such that the image and the co-image are with the same label. From the transitivity of a map, τ is a bijection. Because $\tau Kx = K\tau x$, $x \in \mathcal{X}$, then $\tau\alpha\tau^{-1} = \alpha$ and $\tau\beta\tau^{-1} = \beta$. And in the Proceeding, for labelling a face $(x)_{\mathcal{P}\gamma}$, $\tau(x)_{\mathcal{P}\gamma} = (\tau x)_{\mathcal{Q}\gamma}$. From Lemma 7.5, this implies that $\tau\mathcal{P}\gamma\tau^{-1} = \mathcal{Q}\gamma$. Based on Theorem 7.2, τ is an isomorphism between P and Q. This is the first statement.

By contradiction to prove the second statement. Assume that there is an isomorphism τ between P and Q. If $\tau(x) = y$, then by Algorithm 7.2 the procedure should terminate at Halt (1). However, a termination at Halt (2) shows that for any $x \in \mathcal{X}$, there is no elements in $\mathcal{Y}$ corresponding to x in an isomorphism between maps P and Q, and hence it is impossible to terminate at Halt (1). This is a contradiction.

Therefore, the theorem is true. $\square$

If missing what is related to y in Algorithm 7.1 and Algorithm 7.2, then for any map $M = (\mathcal{X}, \mathcal{P})$, the procedures will always terminate at Halt (1). Thus, their outputs S and T are, respectively, called a *primal trail code* and a *dual trail code* of M. When an element x and a map P should be indicated, they are denoted by respective $S_x(P)$ and $T_x(P)$.

Theorem 7.5 Let $P = (\mathcal{X}, \mathcal{P})$ and $Q = (\mathcal{Y}, \mathcal{Q})$ be two given maps. Then, they are isomorphic if, and only if, for any $x \in \mathcal{X}$ chosen, there exists an element $y \in \mathcal{Y}$ such that $S_x(P) = S_y(Q)$, or

$T_x(P) = T_y(Q)$.

Proof Necessity. Suppose τ is an isomorphism between maps $P = (\mathcal{X}, \mathcal{P})$ and $Q = (\mathcal{Y}, \mathcal{Q})$. For the given element $x \in \mathcal{X}$, let $y = \tau(x)$. From Theorem 7.3, or Theorem 7.4, $S_x(P) = S_y(Q)$, or $T_x(P) = T_y(Q)$.

Sufficiency. From Theorem 7.3, or Theorem 7.4, it is known that by Algorithm 7.1, or Algorithm 7.2, their outputs induces an isomorphism between $P = (\mathcal{X}, \mathcal{P})$ and $Q = (\mathcal{Y}, \mathcal{Q})$. $\square$

Note 7.3 In justifying whether, or not, two maps are isomorphic, the initial element x can be chosen arbitrarily in one of the two maps to see if there is an element y in the other such that $S_x(P) = S_y(Q)$, or $T_x(P) = T_y(Q)$. This enables us to do for some convenience.

In addition, based on Theorem 7.5, all isomorphisms between two maps can be found if any.

VII.5 Pattern examples

Here, two pattern examples are provided for further understanding the procedures of the two algorithms described in the last section.

Pattern 7.1 Justify whether, or not, two maps $M_1 = (\mathcal{X}_1, \mathcal{P}_1)$ and $M_2 = (\mathcal{X}_2, \mathcal{P}_2)$ are isomorphic where

$$\mathcal{X}_1 = Kx_1 + Ky_1,\ \mathcal{P}_1 = (x_1, y_1, \beta y_1)(\gamma x_1)$$

and

$$\mathcal{X}_2 = Kx_2 + Ky_2,\ \mathcal{P}_2 = (y_2, x_2, \beta y_2)(\gamma x_2).$$

First, for M_1, choose $x = x_1$. By Algorithm 7.1, find $S_x(M_1)$. Let

$$\mathcal{P}_1 = (x_1, y_1, \beta y_1)(\gamma x_1) = uv.$$

Initiation

$$x_1 = 1,\ Kx_1 = \{1, \alpha 1, \beta 1, \gamma 1\},\ u = 0, v = 1,$$
$$S = \emptyset,\ l = 0,\ m = 1.$$

Proceeding

Step 1 $\mathcal{P}_1 = (1, y_1, \beta y_1)(\gamma 1)$.

$$y_1 = 2,\ Ky_1 = \{2, \alpha 2, \beta 2, \gamma 2\},\ u = 0,\ v = 1,$$
$$S = \langle 1, 2, \beta 2\rangle,\ l = 2,\ n = 1,\ m = 1.$$

Step 2 $\mathcal{P}_1 = (1, 2, \beta 2)(\gamma 1)$.

$$u = 0,\ v = 1,$$
$$S = \langle 1, 2, \beta 2, \gamma 1\rangle,\ l = 2,\ n = 1,\ m = 1.$$

Halt (1) Output: $S_x(M_1) = S = \langle 1, 2, \beta 2, \gamma 1\rangle$.

Then, for M_2, because a link should correspond to a link and a vertex should correspond to a vertex with the same valency, y has only two possibilities for choice, *i.e.*, x_2 and αx_2. Choose $y = x_2$. By Algorithm 7.1, find $S_y(M_2)$. Let

$$\mathcal{P}_2 = (y_2, x_2, \beta y_2)(\gamma x_2) = uv.$$

Initiation

$$x_2 = 1,\ Kx_2 = \{1, \alpha 1, \beta 1, \gamma 1\},\ u = 0, v = 1,$$
$$S = \emptyset,\ l = 0,\ m = 1.$$

Proceeding

Step 1 $\mathcal{P}_2 = (y_1, 1, \beta y_2)(\gamma 1)$.

$$\beta y_2 = 2,\ K\beta y_2 = \{2, \alpha 2, \beta 2, \gamma 2\},\ u = 0,\ v = 1,$$
$$S = \langle 1, 2, \beta 2\rangle,\ l = 2,\ n = 1,\ m = 1.$$

Step 2 $\mathcal{P}_2 = (2, 1, \beta 2)(\gamma 1)$.

$$u = 0,\ v = 1,$$
$$S = \langle 1, 2, \beta 2, \gamma 1\rangle,\ l = 2,\ n = 1,\ m = 1.$$

Halt (1) Output: $S_y(M_2) = S = \langle 1, 2, \beta 2, \gamma 1 \rangle$.

Since $S_x(M_1) = S_y(M_2)$ and $y = x_2$, an isomorphism from M_1 to M_2 is found as τ_1:

$$\tau_1 K x_1 = K x_2,\ \tau_1 K y_1 = K y_2.$$

Then, choose $y = \alpha x_2$. By Algorithm 7.1, find $S_y(M_2)$. Let

$$\mathcal{P}_2 = (\alpha x_2, \alpha y_2, \gamma y_2)(\beta x_2) = uv.$$

Initiation

$\alpha x_2 = 1,\ K\alpha x_2 = \{1, \alpha 1, \beta 1, \gamma 1\},\ u = 0, v = 1,$
$S = \emptyset,\ l = 0,\ m = 1.$

Proceeding

Step 1 $\mathcal{P}_2 = (1, \alpha y_2, \gamma y_2)(\gamma 1)$.

$\alpha y_2 = 2,\ K\alpha y_2 = \{2, \alpha 2, \beta 2, \gamma 2\},\ u = 0,\ v = 1,$
$S = \langle 1, 2, \beta 2 \rangle,\ l = 2,\ n = 1,\ m = 1.$

Step 2 $\mathcal{P}_2 = (1, 2, \beta 2)(\gamma 1)$.

$u = 0,\ v = 1,$
$S = \langle 1, 2, \beta 2, \gamma 1 \rangle,\ l = 2,\ n = 1,\ m = 1.$

Halt (1) Output: $S_y(M_2) = S = \langle 1, 2, \beta 2, \gamma 1 \rangle$.

Since $S_x(M_1) = S_y(M_2)$ and $y = \alpha x_2$, an isomorphism from M_1 to M_2 is found as τ_2:

$$\tau_2 K x_1 = K\alpha x_2,\ \tau_2 K y_1 = K\alpha y_2.$$

In consequence, there are two isomorphisms between M_1 and M_2 above in all. Since $2 \in S_x(M_1)$ but $\gamma 2 \notin S_x(M_1)$, by Lemma 7.4(iii), M_1, M_2 as well, is nonorientable.

Pattern 7.2 Justify whether, or not, $M_1 = (\mathcal{X}_1, \mathcal{P}_1)$ and $M_2 = (\mathcal{X}_2, \mathcal{P}_2)$ are isomorphic where

$$\mathcal{X}_1 = K x_1 + K y_1,\ \mathcal{P}_1 = (x_1, y_1, \gamma y_1)(\gamma x_1)$$

and

$$\mathcal{X}_2 = Kx_2 + Ky_2,\ \mathcal{P}_2 = (y_2, x_2, \gamma y_2)(\gamma x_2).$$

First, for M_1, choose $x = x_1$. By Algorithm 7.2, find $T_x(M_1)$. Let

$$\mathcal{P}_1\gamma = (x_1, \gamma x_1, y_1)(\gamma y_1) = fg.$$

Initiation

$$x_1 = 1,\ Kx_1 = \{1, \alpha 1, \beta 1, \gamma 1\},\ f = 0, g = 1,$$
$$T = \emptyset,\ l = 0,\ m = 0.$$

Proceeding

Step 1 $\mathcal{P}_1\gamma = (1, \gamma 1, y_1)(\gamma y_1)$.

$$y_1 = 2,\ Ky_1 = \{2, \alpha 2, \beta 2, \gamma 2\},\ f = 0,\ g = 1,$$
$$T = \langle 1, \gamma 1, 2\rangle,\ l = 2,\ n = 1,\ m = 1.$$

Step 2 $\mathcal{P}_1\gamma = (1, \gamma 1, 2)(\gamma 2)$.

$$f = 0,\ g = 1,$$
$$T = \langle 1, \gamma 1, 2, \gamma 2\rangle,\ l = 2,\ n = 1,\ m = 1.$$

Halt (1) Output: $T_x(M_1) = T = \langle 1, \gamma 1, 2, \gamma 2\rangle$.

Then, for M_2, because a link should be corresponding to a link and a vertex should be corresponding to a vertex with the same valency, y only has two possibilities for choosing, *i.e.* , x_2 and αx_2. Choose $y = x_2$. By Algorithm 7.2, find $T_y(M_2)$. Let

$$\mathcal{P}_2\gamma = (x_2, \gamma x_2, \gamma y_2)(y_2) = fg.$$

Initiation

$$x_2 = 1,\ Kx_2 = \{1, \alpha 1, \beta 1, \gamma 1\},\ f = 0, g = 1,$$
$$T = \emptyset,\ l = 0,\ m = 0.$$

Proceeding

Step 1 $\mathcal{P}_2\gamma = (1, \gamma 1, \gamma y_2)(y_2)$.

$$\gamma y_2 = 2,\ K\gamma y_2 = \{2, \alpha 2, \beta 2, \gamma 2\},\ f = 0,\ g = 1,$$
$$T = \langle 1, \gamma 1, 2\rangle,\ l = 2,\ n = 1,\ m = 1.$$

Step 2 $\mathcal{P}_2\gamma = (1, \gamma 1, 2)(\gamma 2)$.

$f = 0,\ g = 1,$
$T = \langle 1, \gamma 1, 2, \gamma 2\rangle,\ l = 2,\ n = 1,\ m = 1.$

Halt (1) Output: $T_y(M_2) = T = \langle 1, \gamma 1, 2, \gamma 2\rangle$.

Since $T_x(M_1) = T_y(M_2)$ and $y = x_2$, an isomorphism from M_1 to M_2 is found as τ_1:

$$\tau_1 K x_1 = K x_2,\ \tau_1 K y_1 = K\gamma y_2.$$

Then, choose $y = \alpha x_2$. By Algorithm 7.2, find $T_y(M_2)$. Let

$$\mathcal{P}_2\gamma = (\alpha x_2, \beta x_2, \alpha y_2)(\beta y_2) = fg.$$

Initiation

$\alpha x_2 = 1,\ K\alpha x_2 = \{1, \alpha 1, \beta 1, \gamma 1\},\ f = 0, g = 1,$
$T = \emptyset,\ l = 0,\ m = 0.$

Proceeding

Step 1 $\mathcal{P}_2\gamma = (1, \gamma 1, \alpha y_2)(\beta y_2)$.

$\alpha y_2 = 2,\ K\gamma y_2 = \{2, \alpha 2, \beta 2, \gamma 2\},\ f = 0,\ g = 1,$
$T = \langle 1, \gamma 1, 2\rangle,\ l = 2,\ n = 1,\ m = 1.$

Step 2 $\mathcal{P}_2\gamma = (1, \gamma 1, 2)(\gamma 2)$.

$f = 0,\ g = 1,$
$T = \langle 1, \gamma 1, 2, \gamma 2\rangle,\ l = 2,\ n = 1,\ m = 1.$

Halt (1) Output: $T_y(M_2) = T = \langle 1, \gamma 1, 2, \gamma 2\rangle$.

Since $T_x(M_1) = T_y(M_2)$ and $y = \alpha x_2$, an isomorphism from M_1 to M_2 is found as τ_2:

$$\tau_2 K x_1 = K\alpha x_2,\ \tau_2 K y_1 = K\alpha y_2.$$

In consequence, there are two isomorphisms between M_1 and M_2 in all. By Lemma 7.4(iii), M_1, M_2 as well, is orientable.

Activities on Chapter VII

VII.6 Observations

O7.1 Observe that whether, or not, an isomorphism between two maps is always mapping a link to a link and a loop to a loop. If it is, describe the reason. Otherwise, by an example.

O7.2 Observe that whether, or not, an isomorphism between two maps is always mapping an element incident with a vertex of valency i to an element incident with a vertex of valency i. If it is, describe the reason. Otherwise, by an example.

O7.3 If missing $\tau\alpha\tau^{-1} = \alpha$ or $\tau\beta\tau^{-1} = \beta$ in (7.2), whether, or not, τ is still an isomorphism. If it is, describe the reason. Otherwise, by an example.

O7.4 Provide two distinct embeddings of a graph which are two isomorphic maps.

O7.5 Observe that how many non-isomorphic maps among all embeddings of the complete graph of order 4.

O7.6 List all non-isomorphic maps of size 3 and find the distribution by relative genus.

O7.7 Explain the differences between non-isomorphic maps with the same under graph and distinct embeddings of the graph by examples.

O7.8 Explain the differences between non-isomorphic graphs and distinct embeddings of these graphs by examples.

O7.9 Let τ_1 and τ_2 are two isomorphisms between two maps. Observe whether, or not, their composition $\tau_1\tau_2^{-1}\tau_1$ is also an isomorphism. If it is, describe the reason. Otherwise, by an example.

O7.10 Observe some algebraic properties on the set of all isomorphisms between two maps.

O7.11 Observe that for two maps $M_1 = (\mathcal{X}_1, \mathcal{P}_1)$ and $M_2 = (\mathcal{X}_1, \mathcal{P}_2)$, is the composition of two permutations $\mathcal{P}_1$ and $\mathcal{P}_2$ a map? If it is, describe the reason. Otherwise, by an example.

O7.12 Observe some algebraic properties of the sets of all maps and premaps on the same ground set under the composition of permutations.

VII.7 Exercises

If an edge is with its two ends of valencies i and j, then it is called a (i,j)-*edge*, $0 \leq i,j \leq 2\epsilon$. If its two incident faces are of valencies l and s, then it is called a $(l,s)^*$-*edge*, $0 \leq s,t \leq 2\epsilon$. Here, ϵ is the size of a map.

E7.1 Let $m_{ij}(M)$ and $n_{ij}(M)$ are, respectively, the numbers of (i,j)-edge and $(i,j)^*$-edge in map M. Prove that if maps M_1 and M_2 are isomorphic, then for any i and j, $0 \leq i,j \leq 2\epsilon$, $m_{ij}(M_1) = m_{ij}(M_2)$ and $n_{ij}(M_1) = n_{ij}(M_2)$.

E7.2 Prove that a bijection between the basic sets of two maps is an isomorphism of the two maps if, and only if, it induces both the correspondences between their vertices and between their faces.

E7.3 Design an algorithm which is different from Algorithm 7.1 and Algorithm 7.2 for justifying an isomorphism between two maps such that its computation amount is in the same order as their's.

E7.4 Prove that Algorithm 7.1 and Algorithm 7.2 are with the computation order $O(\epsilon)$ in justifying an isomorphism of two maps which have a triangular face, or a vertex of valency 3 where ϵ is the

size of the maps.

E7.5 Prove that Algorithm 7.1 and Algorithm 7.2 are with the computation order $O(\epsilon)$ in justifying an isomorphism of two maps which have only one —*articulate vertex*(a vertex of valency 1), or only one loop where ϵ is the size of the maps.

E7.6 Determine the number of non-isomorphic butterflies of size $m \geq 1$, or establish a method to list them.

E7.7 Determine the number of non-isomorphic barflies of size $m \geq 1$, or establish a method to list them.

E7.8 On the basis of Algorithm 7.1, design an algorithm for justifying an isomorphism between two planar graphs(not maps!), and estimate its computation order.

E7.9 On the basis of Algorithm 7.2, design an algorithm for justifying an isomorphism between two planar graphs(not maps!), and estimate its computation order.

For any map M, let T be a spanning tree of its under graph $G(M)$. Each co-tree edge is partitioned into two semi-edges seen as edges. Because what is obtained is just a tree when each of such semi-edges is seen with a new articulate vertex, it is a joint tree corresponding to an embedding of its under graph $G(M)$, also called an *joint tree* of M.

If x and βx are in the same direction for an edge $X = Kx$ partitioned along the joint tree, then it is said to be with the *same sign*, denoted by X and X; otherwise, *different signs*, denoted by X and X^{-1}, or X^{-1} and X. The cyclic order of letters with signs of such semi-edges partitioned into is called an *joint sequence* of the map.

E7.10 Prove that a graph G is planar if, and only if, there exists an joint sequence of maps whose under graph is G such that each letter is with different signs and no two letters are interlaced.

E7.11 Prove that for any complete graph K_n, $n \geq 1$, there is no joint tree for all maps whose under graph are K_n such that it corresponds to a simplified butterfly.

VII.8 Researches

R7.1 Discuss whether, or not, there exits a number, independent on the size of a map considered, of invariants within isomorphism of maps for justifying and recognizing an isomorphism between two maps.

R7.2 For a given graph G and an integer g, determine the number of distinct embeddings of G on the surface of relative genus g, and the number of non-isomorphic maps among them.

R7.3 For a given type of graphs $\mathcal{G}$ and an integer g, find the number of distinct embeddings of graphs in $\mathcal{G}$ on the surface of relative genus g, and the number of non-isomorphic maps among them.

R7.4 Determine the number of non-isomorphic triangulations of size $m \geq 3$.

R7.5 Determine the number of non-isomorphic quadrangulations of size $m \geq 4$.

R7.6 For an integral vector $(n_2, n_4, \cdots, n_{2i}, \cdots)$, find the number of non-isomorphic Euler planar maps each of which has n_{2i} vertices of valency $2i$, $i \geq 1$.

Because it can be shown that two graphs G_1 and G_2 are isomorphic if, and only if, for a surface they can be embedded into, there exist embeddings $\mu_1(G_1)$ and $\mu_2(G_2)$ isomorphic, this enables us to investigate the isomorphism between two graphs. The aim is at an efficient algorithm if any.

R7.7 Suppose map M_1 is an embedding of G_1 on an orientable surface of genus g, justify whether, or not, there is an embedding M_2 of graph G_2 such that M_2 and M_1 are isomorphic.

R7.8 Suppose map M_1 is an embedding of G_1 on a non-orientable surface of genus g, justify whether, or not, there is an embedding M_2 of graph G_2 such that M_2 and M_1 are isomorphic.

R7.9 According to [Liu1], any graph with at least a circuit has a non-orientable embedding with only one face. Justify whether, or

not, two graphs G_1 and G_2 have two respective single face embeddings which are isomorphic.

R7.10 Justify whether, or not, a graph has two distinct single face embeddings which are isomorphic maps.

A graph is called *up-embeddable* if it has an orientable embedding of genus which is the integral part of half the Betti number of the graph. Because of the result in [Liu1], unnecessary to consider the up-embeddability for non-orientable case.

R7.11 Determine the up-embeddability and the maximum orientable genus of a graph via its joint sequences.

R7.12 For a given graph G and an integer g, justify whether, or not, the graph G has an embedding of relative genus g.

Chapter VIII

Asymmetrization

- An automorphism of a map is an isomorphism from the map to itself. All automorphisms of a map form a group called its automorphism group. of a map is, in fact, the trivialization of its automorphism group.
- A number of sharp upper bounds of automorphism group orders for a variety of maps are provided.
- The automorphism groups of simplified butterflies and those of simplified barflies are determined.
- The realization of of a map is from rooting an element of the ground set.

VIII.1 Automorphisms

An isomorphism of a map to itself is called an *automorphism* . Let τ be an automorphism of map $M = (\mathcal{X}, \mathcal{P})$. If for $x \in \mathcal{X}$, $\tau(x) = y$ and $x \neq y$, then two elements x and y play the same role on M, or say, they are *symmetric*. Hence, an automorphism of a map reflects the symmetry among elements in the ground set of the map.

Lemma 8.1 Suppose τ_1 and τ_2 are two automorphisms of map M, then their composition $\tau_1\tau_2$ is also an automorphism of map M.

Proof Because τ_1 is an automorphism of $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, from (7.2)

$$\tau_1\alpha\tau_1^{-1} = \alpha,\ \tau_1\beta\tau_1^{-1} = \beta,\ \tau_1\mathcal{P}\tau_1^{-1} = \mathcal{P}.$$

Similarly, for τ_2,

$$\tau_2\alpha\tau_2^{-1} = \alpha,\ \tau_2\beta\tau_2^{-1} = \beta,\ \tau_2\mathcal{P}\tau_2^{-1} = \mathcal{P}.$$

Therefore, for $\tau_1\tau_2$,

$$\begin{aligned}(\tau_1\tau_2)\alpha(\tau_1\tau_2)^{-1} &= (\tau_1\tau_2)\alpha(\tau_2^{-1}\tau_1^{-1})\\ &= \tau_1(\tau_2\alpha\tau_2^{-1})\tau_1^{-1}\\ &= \tau_1\alpha\tau_1 = \alpha,\end{aligned}$$

$$\begin{aligned}(\tau_1\tau_2)\beta(\tau_1\tau_2)^{-1} &= (\tau_1\tau_2)\beta(\tau_2^{-1}\tau_1^{-1})\\ &= \tau_1(\tau_2\beta\tau_2^{-1})\tau_1^{-1}\\ &= \tau_1\beta\tau_1 = \beta,\end{aligned}$$

and

$$\begin{aligned}(\tau_1\tau_2)\mathcal{P}(\tau_1\tau_2)^{-1} &= (\tau_1\tau_2)\mathcal{P}(\tau_2^{-1}\tau_1^{-1})\\ &= \tau_1(\tau_2\mathcal{P}\tau_2^{-1})\tau_1^{-1}\\ &= \tau_1\mathcal{P}\tau_1 = \mathcal{P}.\end{aligned}$$

This implies that for $\tau_1\tau_2$, (7.2) is commutative. From Theorem 7.1, $\tau_1\tau_2$ is an automorphism of M as well. $\square$

On the basis of the property on permutation composition, automorphisms satisfy the associate law for composition.

Because an automorphism τ is a bijection, it has a unique inverse denoted by τ^{-1}. Because

$$\tau^{-1}\alpha\tau = \tau^{-1}(\tau\alpha\tau^{-1})\tau = (\tau^{-1}\tau)\alpha(\tau^{-1}\tau) = \alpha,$$

and similarly,

$$\tau^{-1}\beta\tau = \beta,\ \tau^{-1}\mathcal{P}\tau = \mathcal{P},$$

from Theorem 7.1, τ^{-1} is also an automorphism.

If an element $x \in \mathcal{X}$ has $\tau(x) = x$ for a mapping(particularly, an automorphism) τ, then x is called a *fixed point* of τ. If every element

is a fixed point of τ, then τ is called an *identity* . Easy to see that an identity on $\mathcal{X}$ is, of course, an automorphism of M, usually said to be *trivial* . By the property of a permutation, an identity is the unity of automorphisms, always denoted by 1.

In summary, the set of all automorphisms of a map M forms a group, called the *automorphism group* of M, denoted by $\mathrm{Aut}(M)$. Its *order* is the cardinality of the set $\mathrm{aut}(M) = |\mathrm{Aut}(M)|$, *i.e.*, the number of elements in $\mathrm{Aut}(M)$ because of the finiteness.

Theorem 8.1 Let τ be an automorphism of map $M = (\mathcal{X}, \mathcal{P})$. If τ has a fixed point, the $\tau = 1$, *i.e.*, the identity.

Proof Suppose x is the fixed point, *i.e.* , $\tau(x) = x$. Because τ is an isomorphism, From (7.1),

$$\begin{aligned}\tau(\alpha x) &= \alpha\tau(x) = \alpha x,\\ \tau(\beta x) &= \beta\tau(x) = \beta x\\ \tau(\mathcal{P}x) &= \mathcal{P}(\tau(x)) = \mathcal{P}x,\end{aligned}$$

i.e., αx, βx and $\mathcal{P}x$ are all fixed points.

Then for any $\psi \in \Psi_{\{\alpha,\beta,\mathcal{P}\}}$,

$$\tau(\psi(x)) = \psi(\tau(x)) = \psi(x).$$

Therefore, from transitive axiom, every element on $\mathcal{X}$ is a fixed point of τ. This means that τ is the identity. $\square$

In virtue of this theorem, the automorphism induced from $\tau(x) = y$ can be represented by $\tau = (x \to y)$.

Example 8.1 Let us go back to the automorphisms of the maps described in Pattern 7.1 and Pattern 7.2.

If M_1 and M_2 in Pattern 7.1 are taken to be

$$M = (Kx + Ky, (x, y, \beta y)(\gamma x)) = M_1,$$

then it is seen that only one nontrivial automorphism $\tau = (x \to \alpha x)$ exists. Thus, its automorphism group is

$$\mathrm{Aut(M)} = \{1, (\mathrm{x} \to \alpha\mathrm{x})\},$$

i.e., a group of order 2.

Then, maps M_1 and M_2 in Pattern 7.2 are taken to be

$$M = (Kx + Ky, (x, y, \gamma y)(\gamma x)) = M_2,$$

it has also only one nontrivial automorphism $\tau = (x \rightarrow \alpha x)$. So, its automorphism group is

$$\mathrm{Aut(M)} = \{1, (\mathrm{x} \rightarrow \alpha \mathrm{x})\},$$

a group of order 2, as well.

However, maps M_1 and M_2 here are not isomorphic. In fact, it is seen that M_1 is nonorientable with relative genus -1. and M_2 is orientable of relative genus 1.

VIII.2 Upper bounds of group order

Because the automorphism group of a combinatorial structure with finite elements is an finite permutation group in its own right, its order must be bounded by an finite number. And, because there are $n!$ permutations on a combinatorial structure of n elements, the order of its automorphism group is bounded by $n!$.

However, $n!$ is an exponential function of n according to the Stirling approximate formula, it is too large for determining the automorphism group in general.

Now, it is asked that is there an constant c such that the order of automorphism group is bounded by n^c, or denoted by $O(n^c)$, if there is, then such a result would be much hopeful for the determination of the group efficiently.

In matter of fact, if the order of automorphism group is $O(n^c)$, c is independent of n for a structure with n elements, then an efficient algorithm can be designed for justifying and recognizing if two of them are isomorphic in a theoretical point of view.

Lemma 8.2 For any map $M = (\mathcal{X}, \mathcal{P})$, the order of its automorphism group is

$$\mathrm{aut}(M) \leq |\mathcal{X}| = 4\epsilon(M) \tag{8.1}$$

where $\epsilon(M) = \frac{1}{4}|\mathcal{X}|$ is the size of M.

Proof From Theorem 8.1, M has at most $|\mathcal{X}| = 4\epsilon(M)$ automorphism, *i.e.*, (8.1). □

The bound presented by this lemma is *sharp*, *i.e.*, it can not be reduced any more. For an example, the *link map* $L = (Kx, (x)(\gamma x))$. The order of its automorphism group is $4 = |Kx| = \epsilon(L)$.

Lemma 8.3 For an integer $i \geq 1$, let $\nu_i(M)$ be the number of i-vertices(vertex incident with i semi-edges) in map $M = (\mathcal{X}, \mathcal{P})$, then

$$\mathrm{aut}(M) \mid 2i\nu_i(M), \tag{8.2}$$

i.e., $\mathrm{aut}(M)$ is a factor of $2i\nu_i(M)$.

Proof Let $\tau \in \mathrm{Aut}(M)$ be an automorphism of M. For $x \in \mathcal{X}$, $(x)_\mathcal{P}$ is an i-vertex, assume $\tau(x) = y$. From the third relation of (7.1), $(y)_\mathcal{P}$ is also an i-vertex. Then, the elements of $\mathcal{X}_i = \{x|\forall x \in \mathcal{X}, |\{x\}_\mathcal{P}| = i\}$ can be classified by the equivalent relation

$$x \sim_{\mathrm{Aut}} y \Longleftrightarrow \exists \tau \in \mathrm{Aut}(M) x = \tau y$$

induced from the group $\mathrm{Aut}(M)$.

From Theorem 8.1, $\mathrm{Aut}(G)$ has a bijection with every equivalent class. This implies that each class has $\mathrm{aut}(M)$ elements. Therefore,

$$\mathrm{aut}(M) \mid |\mathcal{X}_i|.$$

Because $|\mathcal{X}_i| = 2i\nu_i(M)$, (8.2) is soon obtained. □

This lemma allows to improve, even apparently improve the bound presented by Lemma 8.1 for a map not vertex-regular(each vertex has the same valency).

Lemma 8.4 For an integer $j \geq 1$, let $\phi_j(M)$ be the number of j-faces of map $M = (\mathcal{X}, \mathcal{P})$, then

$$\mathrm{aut}(M) \mid 2j\phi_j(M), \tag{8.3}$$

i.e., $\mathrm{aut}(M)$ is a factor of $2j\phi_j(M)$.

Proof Let $\tau \in \mathrm{Aut}(M)$ be an automorphism of M. For $x \in \mathcal{X}$, $(x)_{\mathcal{P}\gamma}$ is a j-face, assume $\tau(x) = y$. From the first two relations of (7.1), $\tau(\gamma x) = \gamma y$. Then from this and the third relations, $\tau((\mathcal{P}\gamma)x) = (\mathcal{P}\gamma)y$. Thus, $(y)_{\mathcal{P}\gamma}$ is also a j-face. And, the elements of $\mathcal{X}_j = \{x|\forall x \in \mathcal{X}, |\{x\}_{\mathcal{P}\gamma}| = j\}$ can be classified by the equivalence

$$x \sim_{\mathrm{Aut}} y \Longleftrightarrow \exists \tau \in \mathrm{Aut}(M) x = \tau y$$

induced from the group $\mathrm{Aut}(M)$.

Further, from Theorem 8.1, $\mathrm{Aut}(G)$ has a bijection with every equivalent class. This leads that each class has $\mathrm{aut}(M)$ elements. Therefore,

$$\mathrm{aut}(M) \mid |\mathcal{X}_j|.$$

Because $|\mathcal{X}_j| = 2j\phi_j(M)$, (8.3) is soon obtained. □

This lemma allows also to improve, even apparently improve the bound presented by Lemma 8.1 for a map not face-regular(each face has the same valency).

Theorem 8.2 Let $\nu_i(M)$ and $\phi_j(M)$ be, respectively, the numbers of i-vertices and j-faces in map $M = (\mathcal{X}, \mathcal{P})$, $i, j \geq 1$, then

$$\mathrm{aut}(M) \mid (2i\nu_i,\ 2j\phi_j \mid \forall i,\ i \geq 1,\ \forall j,\ j \geq 1), \tag{8.4}$$

where $(2i\nu_i,\ 2j\phi_j \mid \forall i,\ i \geq 1,\ \forall j,\ j \geq 1)$ represents the greatest common divisor of all the numbers in the parentheses.

Proof From Lemma 8.3, for any integer $i \geq 1$,

$$\mathrm{aut}(M) \mid 2i\nu_i(M).$$

From Lemma 8.4, for any integer $j \geq 1$,

$$\mathrm{aut}(M) \mid 2i\phi_j(M).$$

By combining the two relations above, (8.4) is soon found. □

Based on this theorem, the following corollary is naturally deduced.

Corollary 8.1 Let $\nu_i(M)$ and $\phi_j(M)$ be, respectively, the numbers of i-vertices and j-faces in map $M = (\mathcal{X}, \mathcal{P})$, $i, j \geq 1$, then

$$\mathrm{aut}(M) \ \leq \ (2i\nu_i, \ 2j\phi_j \mid \forall i, \ i \geq 1, \ \forall j, \ j \geq 1). \tag{8.5}$$

Proof A direct result of (8.4). □

Corollary 8.2 For map $M = (\mathcal{X}, \mathcal{P})$, $\epsilon(M)$ is its size, then

$$\mathrm{aut}(M) \mid 4\epsilon(M). \tag{8.6}$$

Proof Because

$$\begin{aligned} 4\epsilon(M) &= 2\sum_{i\geq 1} i\nu_i(M) = 2\sum_{j\geq 1} j\phi_j(M) \\ &= \sum_{i\geq 1} 2i\nu_i(M) = \sum_{j\geq 1} 2j\phi_j(M), \end{aligned}$$

we have

$$(2i\nu_i, \ 2j\phi_j \mid \forall i, \ i \geq 1, \ \forall j, \ j \geq 1) \mid 4\epsilon(M).$$

Hence, from Theorem 8.2, (8.6) is soon derived. □

VIII.3 Determination of the group

In this section, the automorphism groups of standard maps, *i.e.*, simplified butterflies and simplified barflies, are discussed.

First, observe the orientable case. For

$$O_1 = (\mathcal{X}_1, \mathcal{J}_1) = (Kx_1 + Ky_1, (x_1, y_1, \gamma x_1, \gamma y_1)),$$

by Algorithm 7.1,

$$\begin{aligned}
S_{x_1}(O_1) &= 1, y_1, \gamma 1, \gamma y_1 = 1, 2, \gamma 1, \gamma 2;\\
S_{\alpha x_1}(O_1) &= 1, \beta y_1, \gamma 1, \alpha y_1 = 1, 2, \gamma 1, \gamma 2;\\
S_{\beta x_1}(O_1) &= 1, \alpha y_1, \gamma 1, \beta y_1 = 1, 2, \gamma 1, \gamma 2;\\
S_{\gamma x_1}(O_1) &= 1, \gamma y_1, \gamma 1, y_1 = 1, 2, \gamma 1, \gamma 2;\\
S_{y_1}(O_1) &= 1, \gamma x_1, \gamma 1, x_1 = 1, 2, \gamma 1, \gamma 2;\\
S_{\alpha y_1}(O_1) &= 1, \alpha x_1, \gamma 1, \beta x_1 = 1, 2, \gamma 1, \gamma 2;\\
S_{\beta y_1}(O_1) &= 1, \beta x_1, \gamma 1, \alpha x_1 = 1, 2, \gamma 1, \gamma 2;\\
S_{\gamma y_1}(O_1) &= 1, x_1, \gamma 1, \gamma x_1 = 1, 2, \gamma 1, \gamma 2,
\end{aligned}$$

i.e.,

$$\begin{aligned}
S_{x_1}(O_1) = S_{\alpha x_1}(O_1) = S_{\beta x_1}(O_1) &= S_{\gamma x_1}(O_1) = S_{y_1}(O_1)\\
&= S_{\alpha y_1}(O_1) = S_{\beta y_1}(O_1) = S_{\gamma y_1}(O_1)\\
&= 1, 2, \gamma 1, \gamma 2.
\end{aligned}$$

Thus, O_1 has its automorphism group of order 8, *i.e.*,

$$\mathrm{aut}(O_1) = 4 \times (2 \times 1) = 8.$$

A map with a non-trivial automorphism group is said to be *symmetrical*. If a map with its automorphism group of order 4 times its size, the it is said to be *completely symmetrical*. It can be seen that O_1 is completely symmetrical. However, none of $O_k,\ k \geq 2$, is completely symmetrical although they are all symmetrical.

Theorem 8.3 For simplified butterflies(orientable standard maps) $O_k = (\mathcal{X}_k, \mathcal{J}_k)$, $k \geq 1$, where

$$\mathcal{X}_k = \sum_{i=1}^{k} (Kx_i + Ky_i)$$

and

$$\mathcal{J}_k = (\prod_{i=1}^{k} \langle x_i, y_i, \gamma x_i, \gamma y_i \rangle),$$

we have

$$\mathrm{aut}(O_k) = \begin{cases} 2k, \text{ if } k \geq 2; \\ 8, \text{ if } k = 1. \end{cases} \tag{8.7}$$

Proof From the symmetry between $\langle x_i, y_i, \gamma x_i, \gamma y_i\rangle$, $i \geq 2$, and $\langle x_1, y_1, \gamma x_1, \gamma y_1\rangle$ in $\mathcal{J}_k$, $k \geq 2$, only necessary to calculate $S_{x_1}(O_k)$, $S_{\gamma x_1}(O_k)$, $S_{y_1}(O_k)$, $S_{\gamma y_1}(O_k)$, $S_{\alpha x_1}(O_k)$, $S_{\beta x_1}(O_k)$, $S_{\alpha y_1}(O_k)$, and $S_{\beta y_1}(O_k)$ by Algorithm 7.1.

From Algorithm 7.1,

$$\begin{aligned} S_{x_1}(O_k) &= 1, y_1, \gamma 1, \gamma y_1, \prod_{i=1}^{k}\langle x_i, y_i, \gamma x_i, \gamma y_i\rangle \\ &= 1, 2, \gamma 1, \gamma 2, \prod_{i=2}^{k}\langle (2i-1), 2i, \gamma(2i-1), \gamma 2i\rangle \\ &= \prod_{i=1}^{k}\langle (2i-1), 2i, \gamma(2i-1), \gamma 2i\rangle, \end{aligned}$$

$$\begin{aligned} S_{\alpha x_1}(O_k) &= 1, \alpha(\prod_{i=1}^{k}\langle x_i, y_i, \gamma x_i, \gamma y_i\rangle)^{-1}, \beta y_1, \gamma 1, \alpha y_1 \\ &\neq S_{x_1}(O_k), \end{aligned}$$

$$\begin{aligned} S_{\gamma x_1}(O_k) &= 1, \gamma y_1, \prod_{i=1}^{k}\langle x_i, y_i, \gamma x_i, \gamma y_i\rangle, \gamma 1, y_1 \\ &\neq S_{x_1}(O_k), \end{aligned}$$

$$\begin{aligned} S_{\beta x_1}(O_k) &= 1, \alpha y_1, \gamma 1, \alpha(\prod_{i=1}^{k}\langle x_i, y_i, \gamma x_i, \gamma y_i\rangle)^{-1}, \beta y_1 \\ &\neq S_{x_1}(O_k), \end{aligned}$$

$$\begin{aligned} S_{y_1}(O_k) &= 1, \gamma x_1, \gamma 1, \prod_{i=1}^{k}\langle x_i, y_i, \gamma x_i, \gamma y_i\rangle, x_1 \\ &\neq S_{x_1}(O_k), \end{aligned}$$

$$S_{\alpha y_1}(O_k) = 1, \alpha x_1, \alpha(\prod_{i=1}^{k}\langle x_i, y_i, \gamma x_i, \gamma y_i\rangle)^{-1}, \gamma 1, \beta x_1$$
$$\neq S_{x_1}(O_k),$$

$$S_{\gamma y_1}(O_k) = 1, \prod_{i=1}^{k}\langle x_i, y_i, \gamma x_i, \gamma y_i\rangle, x_1, \gamma 1, \gamma x_1$$
$$\neq S_{x_1}(O_k),$$

$$S_{\beta y_1}(O_k) = 1, \beta x_1, \alpha 1, \alpha x_1, \alpha(\prod_{i=1}^{k}\langle x_i, y_i, \gamma x_i, \gamma y_i\rangle)^{-1}$$
$$= 1, 2.\gamma 1, \gamma 2, \prod_{i=2}^{k}\langle (2i-1), 2i, \gamma(2i-1), \gamma 2i\rangle$$
$$= S_{x_1}(O_k).$$

Because two automorphisms are from $S_{\beta y_1}(O_k) = S_{x_1}(O_k)$, O_k have $2 \times k = 2k$ automorphisms altogether. Hence, when $k \geq 2$,

$$\mathrm{aut}(O_k) = 2k.$$

When $k = 1$, $\mathrm{aut}(O_1) = 8$ is known. □

Then, observe the nonorientable case. for

$$Q_1 = (\mathcal{X}_1, \mathcal{I}_1) = (Kx_1, (x_1, \beta x_1)),$$

by Algorithm 7.1,

$$S_{x_1}(Q_1) = 1, \beta_1;\ S_{\alpha x_1}(Q_1) = 1, \beta 1;$$
$$S_{\beta x_1}(Q_1) = 1, \beta 1;\ S_{\gamma x_1}(Q_1) = 1, \beta 1,$$

i.e., $\mathrm{aut}(Q_1) = 4$.

Theorem 8.4 For simplified barflies $Q_l = (\mathcal{X}_l, \mathcal{I}_l)$, $l \geq 1$, where

$$\mathcal{X}_l = \sum_{i=1}^{l} Kx_i$$

and

$$\mathcal{I}_l = \prod_{i=1}^{l} \langle x_i, \beta x_i \rangle,$$

we have

$$\operatorname{aut}(Q_l) = \begin{cases} 2l, \text{ 当 } l \geq 2; \\ 4, \text{ 当 } l = 1. \end{cases} \tag{8.8}$$

Proof From the symmetry of $\langle x_i, \beta x_i \rangle$, $i \geq 2$, and $\langle x_1, \beta x_1 \rangle$ in $\mathcal{I}_l$, $l \geq 2$, only necessary to calculate

$$S_{x_1}(Q_l), S_{\alpha x_1}(Q_l), S_{\beta x_1}(Q_l) \text{ and } S_{\gamma x_1}(Q_l)$$

by employing Algorithm 7.1.

From Algorithm 7.1,

$$\begin{aligned} S_{x_1}(Q_l) &= 1, \beta 1, \prod_{i=2}^{l} \langle x_i, \beta x_i \rangle \\ &= 1, \beta 1, 2, \beta 2, \prod_{i=3}^{l} \langle x_i, \beta x_i \rangle \\ &= \prod_{i=1}^{l} \langle i, \beta i \rangle, \end{aligned}$$

$$\begin{aligned} S_{\alpha x_1}(Q_l) &= 1, \alpha \Big(\prod_{i=2}^{l} \langle x_i, \beta x_i \rangle \Big)^{-1}, \beta 1 \\ &\neq S_{x_1}(Q_l), \end{aligned}$$

$$\begin{aligned} S_{\beta x_1}(Q_l) &= 1, \prod_{i=2}^{l} \langle x_i, \beta x_i \rangle, \beta 1 \\ &\neq S_{x_1}(Q_l), \end{aligned}$$

$$
\begin{aligned}
S_{\gamma x_1}(Q_l) &= 1, \beta 1, \alpha(\prod_{i=2}^{l}\langle x_i, \beta x_i\rangle)^{-1} \\
&= 1, \beta 1, l, \beta l, \prod_{i=l-1}^{2}\langle \gamma x_i, \alpha x_i\rangle \\
&= \prod_{i=1}^{l}\langle i, \beta i\rangle \\
&= S_{x_1}(Q_l).
\end{aligned}
$$

Because two automorphisms are from $S_{\gamma x_1}(Q_l) = S_{x_1}(Q_l)$, Q_l has $2 \times l = 2l$ automorphisms altogether. Hence, when $l \geq 2$,

$$\operatorname{aut}(Q_l) = 2l.$$

When $l = 1$, $\operatorname{aut}(Q_1) = 4$ is known. □

Similarly, the two theorems can also be proved by employing Algorithm 7.2.

VIII.4 Rootings

For a given map $M = (\mathcal{X}, \mathcal{P})$, if a subset $R \subseteq \mathcal{X}$ is chosen such that an automorphism of M with R fixed, *i.e.*, an element of R does only correspond to an element of R, then M is called a *set rooted* map. The subset R is called the *rooted set* of M, and an element of R is called a *rooted element* .

Theorem 8.5 For a set rooted map $M^{\mathrm{R}} = (\mathcal{X}, \mathcal{P})$, R is the rooted set,

$$\operatorname{aut}(M^{\mathrm{R}}) \mid |R|. \tag{8.9}$$

Proof Assume that all elements in R are partitioned into equivalent classes under the group $\operatorname{Aut}(M^{\mathrm{R}})$. From Theorem 8.1, each class has $\operatorname{aut}(M^{\mathrm{R}})$ elements. Therefore, (8.9) is satisfied. □

Corollary 8.3 For a set rooted map $M^{\mathrm{R}} = (\mathcal{X}, \mathcal{P})$, R is the rooted set,

$$\mathrm{aut}(M^{\mathrm{R}}) \leq |R|. \tag{8.10}$$

Proof A direct result of (8.9). $\square$

For a given map $M = (\mathcal{X}, \mathcal{P})$, if a vertex v_x, $x \in \mathcal{X}$, is chosen such that an automorphism of M has to be with v_x fixed, *i.e.*, an element incident with v_x has to correspond to an element incident with v_x, then M is called a *vertex rooted* map. The vertex v_x is called the *rooted vertex* of M, and an element incident with v_x, *rooted element* .

Corollary 8.4 For a vertex rooted map $M^{\mathrm{vr}} = (\mathcal{X}, \mathcal{P})$, v_x is the rooted vertex,

$$\mathrm{aut}(M^{\mathrm{vr}}) \mid 2|\{x\}_{\mathcal{P}}|. \tag{8.11}$$

Proof This is (8.9) when $R = \{x\}_{\mathcal{P}} \cup \{\alpha x\}_{\mathcal{P}}$. $\square$

For a given map $M = (\mathcal{X}, \mathcal{P})$, if face f_x, $x \in \mathcal{X}$, is chosen such that an automorphism of M has f_x fixed, *i.e.* , an element incident with f_x should be corresponding to an element incident with f_x, then M is said to be a *face rooted* map. The face f_x is called the *rooted face* of M. An element in rooted face is called an *rooted element.*

Corollary 8.5 For a face rooted map $M^{\mathrm{fr}} = (\mathcal{X}, \mathcal{P})$ with rooted face f_x,

$$\mathrm{aut}(M^{\mathrm{fr}}) \mid 2|\{x\}_{\mathcal{P}\gamma}|. \tag{8.12}$$

Proof This is (8.9) when $R = \{x\}_{\mathcal{P}\gamma} \cup \{\alpha x\}_{\mathcal{P}\gamma}$. $\square$

For given map $M = (\mathcal{X}, \mathcal{P})$, if edge e_x, $x \in \mathcal{X}$, is chosen such that an automorphism of M is with e_x fixed, *i.e.*, an element in e_x is always corresponding to an element in e_x, then M is called an *edge rooted* map. Edge e_x is the *rooted edge* of M. An element in the rooted edge is also called a *rooted element.*

Corollary 8.6 For an edge rooted map $M^{\mathrm{er}} = (\mathcal{X}, \mathcal{P})$ with the rooted edge e_x,

$$\mathrm{aut}(M^{\mathrm{er}}) \mid |Kx|. \tag{8.13}$$

Proof The case of (8.9) when $R = Kx$. $\square$

For a given map $M = (\mathcal{X}, \mathcal{P})$, an element $x \in \mathcal{X}$ is chosen such that an automorphism of M is with x as a fixed point, then M is called a *rooted map* . The element x is the *root* of M. The vertex, the edge and the face incident to the root are, respectively, called the *root vertex*, the *root edge* and the *root face*.

Corollary 8.7 For a rooted map $M^{\mathrm{r}} = (\mathcal{X}, \mathcal{P})$ with its root x,

$$\mathrm{aut}(M^{\mathrm{r}}) = 1. \tag{8.14}$$

Proof The case of (8.9) when $R = \{x\}$. $\square$

This tells us that a rooted map does not have the symmetry at all. The way mentioned above shows such a general clue for transforming a problem with symmetry to a problem without symmetry and then doing the reversion.

Example 8.2 Map

$$M_1 = (Kx + Ky, (x)(\gamma x, y, \gamma y))$$

has 4 distinct ways for choosing the root. Because M_1 has the following 4 primal trail codes

$$S_x = \underline{1}_0, \underline{\gamma 1, 2, \gamma 2}_1 = S_{\alpha x}, S_{\gamma x} = \underline{1, 2, \gamma 2}_0, \underline{\gamma 1}_1 = S_{\beta x}$$

$$S_y = \underline{1, \gamma 1, 2}_0, \underline{\gamma 2}_1 = S_{\beta y}, S_{\gamma y} = \underline{1, 2, \gamma 1}_0 \underline{\gamma 2}_1 = S_{\alpha y},$$

the 4 ways of rooting are shown in Fig.8.1(a–d) where the root is marked at its tail.

Example 8.3 Map $M_2 = (Kx + Ky, (x)(\gamma x, y, \beta y))$ has 4 distinct ways for choosing the root. Because M_2 has the following 4

primal trail codes

$$S_x = \underline{1}_0, \underline{\gamma 1, 2, \beta 2}_1 = S_{\alpha x}, S_{\gamma x} = \underline{1, 2, \beta 2}_0, \underline{\gamma 1}_1 = S_{\beta x}$$

$$S_y = \underline{1, \beta 1, 2}_0, \underline{\gamma 2}_1 = S_{\gamma y}, S_{\beta y} = \underline{1, 2, \beta 1}_0 \underline{\gamma 2}_1 = S_{\alpha y},$$

the 4 ways of rooting are shown in Fig.8.2(a–d) where the root is marked at its tail.

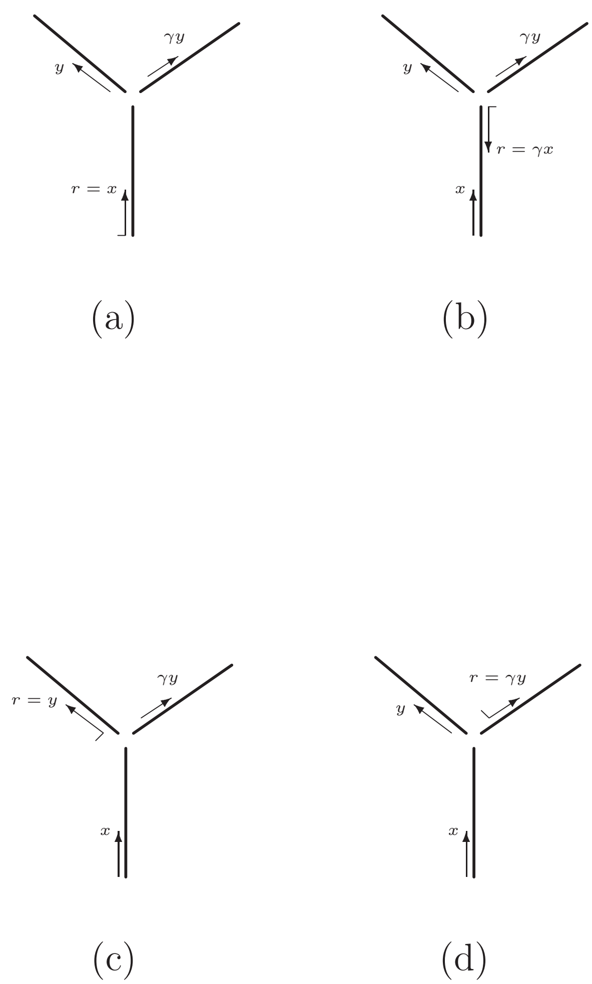


Fig.8.1 Rootings in Example 1

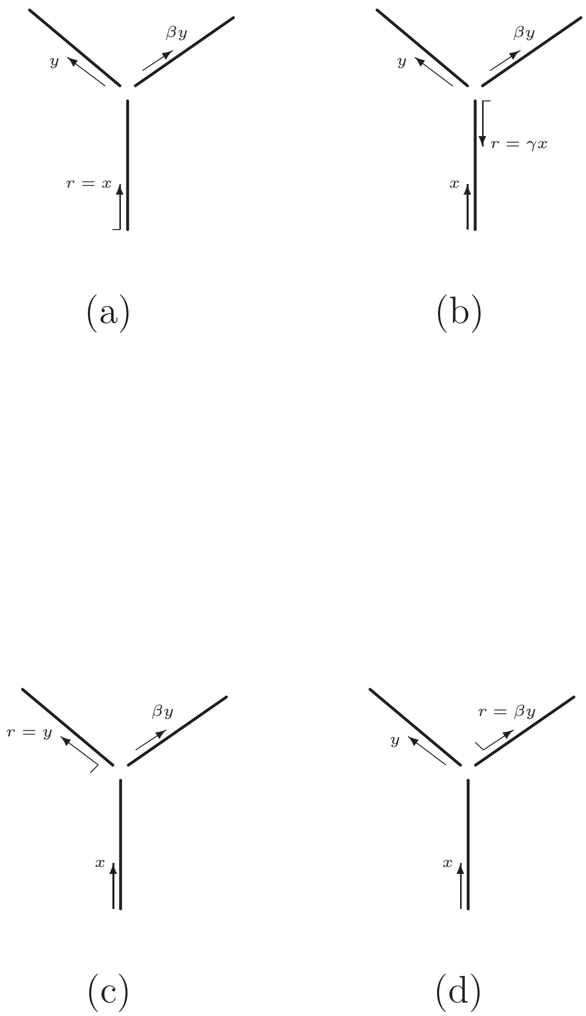


Fig.8.2 Rootings in Example 2

Activities on Chapter VIII

VIII.5 Observations

O8.1 Observe that how many embeddings the complete graph K_4 of order 4 has. How many non-isomorphic maps are there among them?.

O8.2 Consider the automorphism of the cube. Observe that what happens to the automorphism of those obtained by deleting, contracting, splitting and appending an edge on the cube.

O8.3 Consider the automorphism of the octahedron. Observe that what happens to the automorphism of those obtained by deleting, contracting, splitting and appending an edge on the octahedron.

O8.4 Consider the automorphism of the dodecahedron. Observe that what happens to the automorphism of those obtained by deleting, contracting, splitting and appending an edge on the dodecahedron.

O8.5 Consider the automorphism of the icosahedron. Observe that what happens to the automorphism of those obtained by deleting, contracting, splitting and appending an edge on the icosahedron.

O8.6 Observe the duality between O8.2 and O8.3 and between O8.4 and O8.5.

O8.7 Consider how to justify is there an edge in a map so that the order of the automorphism group of what is obtained by deleting the edge on the map does not change.

O8.8 Consider how to justify is there an edge in a map so

that the order of the automorphism group of what is obtained by contracting the edge on the map does not change.

O8.9 Consider how to justify is there an edge in a map so that the order of the automorphism group of what is obtained by splitting the edge on the map does not change.

O8.10 Consider how to justify is there an edge in a map so that the order of the automorphism group of what is obtained by appending the edge on the map does not change.

O8.11 Provide a map whose automorphism group is the cyclic group.

O8.12 List all maps of size 3 with their automorphisms of order 2.

VIII.6 Exercises

From the last two chapters, it is seen that the automorphism of a map provides a foundation for justifying is the map isomorphic to another. This is an pattern example for the automorphism of a general combinatorial structure. For instance, a graph, a network, a combinatorial design, a lattice, a group, a ring, a field *etc.*

E8.1 Observe whether, or not, an automorphism of a map induces an automorphism of its under graph.

E8.2 Determine the automorphism group of the map $(\mathcal{X}, \mathcal{P})$ where

$$\mathcal{X} = \sum_{i=1}^{k}(Kx_i + Ky_i)$$

and

$$\mathcal{P} = (\prod_{i=1}^{k}\langle x_i, y_i\rangle, \prod_{i=1}^{k}\langle \gamma x_i, \gamma y_i\rangle)$$

for $k \geq 2$.

E8.3 Determine the automorphism group of the map $(\mathcal{X}, \mathcal{P})$ where

$$\mathcal{X} = \sum_{i=1}^{k} Kx_i$$

and

$$\mathcal{P} = (x_1, x_2, \cdots, x_k, \beta x_k, \cdots, \beta x_2, \beta x_1)$$

for $k \geq 2$.

A *primal matching* of a map is defined to be such a set of edges that any pair of its edges have no common end. If a primal matching is incident to all the vertices on the map, then it is said to be *perfect* . A *dual matching* of a map is such a set of edges that any pair of its edges have no incident face in common. If a dual matching is incident to all the faces on the map, then it is said to be *perfect* as well. By no means any map has a perfect primal matching. One having a prefect primal matching is called a *primal matching map*. By no means any map has a perfect dual matching either. One having a prefect dual matching is called a *dual matching map*. A map which is both of primal matching and of dual matching is said to be of *bi-matching*.

E8.4 Suppose M is a primal matching map and P, a perfect primal matching of M. Let $n_{i,j}(P)$ be the number of (i, j)-edges in P. Prove that for any i, j, $n_{i,j}(P) \neq 0$,

$$\mathrm{aut}(M) \mid 2(i + j)n_{i,j}(P).$$

E8.5 Suppose M is a dual matching map and P^*, a perfect dual matching. Let $n^*_{i,j}(P^*)$ be the number of $(i, j)^*$-edges in P^*. Prove that for any i, j, $n^*_{i,j}(P^*) \neq 0$,

$$\mathrm{aut}(M) \mid 2(i + j)n^*_{i,j}(P^*).$$

E8.6 Design an algorithm for justifying does a primal matching map, a dual matching map, or a bi-matching map have an non-trivial automorphism. Explain their efficiency. Provide a condition for the

automorphism group of a primal matching map, a dual matching map, or a bi-matching map with, respectively, a primal matching, a dual matching, or a bi-matching rooted as the same as without rooting.

For a map M, if a set of its faces is pairwise without common edge, then it is said to be *independent*. If an independent face set is pairwise without common vertex, it is called a *cavity* . If a cavity is *spanning*, *i.e.*, all vertices of M are incident with the cavity, then it is called a *full cavity* .

E8.7 Recognize if a vertex 3-regular(*i.e.*, *cubic*) map has a full cavity.

E8.8 For a vertex 3-regular map M with a full cavity H, let $h_i(H)$ be the number of i^*-faces, $i \geq 1$, in this cavity. Prove that for any integer $i \geq 1$, $h_i(H) > 0$,

$$\mathrm{aut}(M) \mid 6ih_i(H).$$

E8.9 Recognize whether, or not, a map has a full cavity.

For a face f in a cavity H, its *H-valency* is the number of its incident primal semi-edges except for those in the boundary.

E8.10 For a map M, let $h_i(H;j)$ be the number of j-faces of H-valency i. Prove that

$$\mathrm{aut}(M) \mid 2(i+2j)h_i(H;j).$$

E8.11 For a given integer k, find the number of non-isomorphic butterflies of relative genus $k(k \geq 0)$, or of barflies of relative genus $k(k < 0)$.

VIII.7 Researches

R8.1 Given 3 integers $m \geq 1$, g and $s \geq 1$, determine the

number of primal matching maps of relative genus g with size m and the order s of their automorphism groups.

R8.2 Given 3 integers $m \geq 1$, g and $s \geq 1$, determine the number of dual matching maps of relative genus g with size m and the order s of their automorphism groups.

R8.3 Given 3 integers $m \geq 1$, g and $s \geq 1$, determine the number of bi-matching maps of relative genus g with size m and the order s of their automorphism groups.

The first three problems should be considered for starting from $g = 0$, 1, and then -1. Particularly, the three problems for self-dual maps should be firstly studied before the general cases.

R8.4 Find the cubic maps of size $m \geq 7$ with a given relative genus and the maximum order of their automorphism groups

R8.5 Find the maps of size $m \geq 1$ with a given relative genus and the order 1 of their automorphism groups.

R8.6 Prove, or disprove, the conjecture that for a given relative genus, almost all maps have their automorphism groups of order 1.

R8.7 Given three integers $m \geq 1$, g and $s \geq 1$, determine the full cavity maps of size m with relative genus g and the order of their automorphism groups s.

If a map has a set of edges inducing a Hamiltonian circuit on its under graph, then it is called a *primal* H-*map* . If a map has e set of edges inducing a Hamiltonian circuit on the under graph of its dual, then it is called a *dual* H-*map* . If a map is both a primal H-map and a dual H-map, then it is called a *double* H-*map*.

R8.8 Given three integers $m \geq 1$, g and $s \geq 1$, determine the primal H-maps of size m with relative genus g and their automorphism group of order s.

R8.9 Given three integers $m \geq 1$, g and $s \geq 1$, determine the

dual H-maps of size m with relative genus g and their automorphism group of order s.

R8.10 Given three integers $m \geq 1$, g and $s \geq 1$, determine the double H-maps of size m with relative genus g and their automorphism group of order s.

Chapter IX

Rooted Petal Bundles

- A petal bundle is a map which has only one vertex, or in other words, each edge of self-loop.
- From decomposing the set of rooted orientable petal bundles, a linear differential equation satisfied by the enumerating function with size as the parameter is discovered and then an explicit expression of the function is extracted.
- A quadratic equation of the enumerating function for rooted petal bundles on the surface of orientable genus 0 is discovered and then an explicit expression is extracted.
- From decomposing the set of rooted nonorientable petal bundles, a linear differential equation satisfied by the enumerating function with size as the parameter is discovered in company with the orientable case and then a favorable explicit expression of the function is also extracted.
- The numbers of orientable, nonorientable and total petal bundles with given size are, separately, obtained and then calculated for size not greater than 10.

IX.1 Orientable petal bundles

A single vertex map is also called a *petal bundle*, its under graph is a *bouquet*. In this section, the orientable rooted petal bundles are investigated for determining their enumerating function with size as a parameter by a simple form.

Let $\mathcal{D}$ be the set of all non-isomorphic orientable rooted petal bundles. For convenience, the trivial map ϑ is assumed to be in $\mathcal{D}$.

Now, $\mathcal{D}$ is divided into two classes: $\mathcal{D}_{\mathrm{I}}$ and $\mathcal{D}_{\mathrm{II}}$, *i.e.*,

$$\mathcal{D} = \mathcal{D}_{\mathrm{I}} + \mathcal{D}_{\mathrm{II}} \tag{9.1}$$

where $(\mathcal{D})_{\mathrm{I}} = \{\vartheta\}$ and $\mathcal{D}_{\mathrm{II}}$, of course, consists of all petal bundles in $\mathcal{D}$ other than ϑ.

Lemma 9.1 Let $\mathcal{D}_{\langle \mathrm{II} \rangle} = \{D - a | \forall D \in \mathcal{D}_{\mathrm{II}}\}$. Then,

$$\mathcal{D}_{\langle \mathrm{II} \rangle} = \mathcal{D}. \tag{9.2}$$

Proof For any $D = (\mathcal{X}, \mathcal{P}) \in \mathcal{D}_{\langle \mathrm{II} \rangle}$, there exists a $D' = (\mathcal{X}', \mathcal{P}') \in \mathcal{D}_{\mathrm{II}}$ such that $D = D' - a'$. Because D' is orientable, group $\Psi' = \Psi_{\{\gamma, \mathcal{P}'\}}$ has two orbits

$$\{r'\}_{\mathcal{P}'} \text{ and } \{\alpha r'\}_{\mathcal{P}'}$$

on $\mathcal{X}'$. Because $\gamma r' \in \{r'\}_{\mathcal{P}'}$, D has also two orbits

$$\{r\}_{\mathcal{P}} = \{r'\}_{\mathcal{P}'} - \{r', \gamma r'\} \text{ and } \{\alpha r\}_{\mathcal{P}} = \{\alpha r'\}_{\mathcal{P}'} - \{\alpha r', \beta r'\},$$

and hence D is orientable as well. From Theorem 3.4, petal bundle D' leads that$D$ is a petal bundle. This implies that $\mathcal{D}_{\langle \mathrm{II} \rangle} \subseteq \mathcal{D}$.

Conversely, for any $D = (\mathcal{X}, \mathcal{P}) \in \mathcal{D}$, $\mathcal{P} = (r, \mathcal{P}r, \cdots, \mathcal{P}^{-1}r)$, $D' = (\mathcal{X} + Kr', \mathcal{P}')$ where

$$\mathcal{P}' = (r', \gamma r', r, \mathcal{P}r, \cdots, \mathcal{P}^{-1}r).$$

Because D is orientable, group $\Psi = \Psi_{\{\gamma, \mathcal{P}\}}$ has two orbits $\{r\}_{\Psi}$ and $\{\alpha r\}_{\Psi}$ on $\mathcal{X}$. Thus, group $\Psi' = \Psi_{\{\gamma, \mathcal{P}'\}}$ has two orbits

$$\{r'\}_{\Psi'} = \{r\}_{\Psi} + \{r', \gamma r'\} \text{ and } \{\alpha r'\}_{\Psi'} = \{\alpha r\}_{\Psi} + \{\alpha r', \beta r'\}$$

on $\mathcal{X}'$. This means that D' is also orientable. Because D' has only one vertex, $D' \in \mathcal{D}$. And, from $D' \neq \vartheta$, it is only possible that $D' \in \mathcal{D}_{\mathrm{II}}$. Therefore, in view of $D = D' - a'$, $\mathcal{D} \subseteq \mathcal{D}_{\langle \mathrm{II} \rangle}$. □

From the last part in the proof of this lemma, for any $D = (r)_{\mathcal{J}} \in \mathcal{D}$, D' has $2m(D)+1$ distinct choices such that $D' = D_i = D + e_i \in \mathcal{D}_{\mathrm{II}}$, $0 \leq i \leq 2m(D)$, and hence $D = D' - a'$ where $e_i = Kr'$ and

$$\begin{cases} D_0 = (r', \gamma r', r, \mathcal{J}r, \cdots, \mathcal{J}^{2m(D)-1}r), \ i = 0; \\ D_i = (r', r, \cdots, \mathcal{J}^{i-1}r, \gamma r', \mathcal{J}^i r, \cdots, \mathcal{J}^{2m(D)-1}), \\ \qquad\quad 1 \leq i \leq 2m(D) - 1; \\ D_{2m(D)} = (r', r, \mathcal{J}r, \cdots, \mathcal{J}^{2m(D)-1}r, \gamma r'), i = 2m(D), \end{cases}$$

for $\gamma = \alpha\beta$.

Lemma 9.2 Let $\mathcal{H}(D) = \{D_i | i = 0, 1, 2, \cdots, 2m(D)\}$ for $D \in \mathcal{D}$. Then,

$$\mathcal{D}_{\mathrm{II}} = \sum_{D \in \mathcal{D}} \mathcal{H}(D). \tag{9.3}$$

Proof Because of Lemma 9.1, it is easily seen that the set on the left hand side of (9.3) is a subset of the set on the right.

Conversely, from $\mathcal{H}(D) \subseteq \mathcal{D}_{\mathrm{II}}$, for any $D \in \mathcal{D}$, the set on the right hand side of (9.3) is also a subset on the left. □

The importance of Lemma 9.2 is that (9.3) provides a 1–to–1 correspondence between the sets on its two sides. This is seen from the fact that for any two non-isomorphic petal bundles D_1 and D_2, $\mathcal{H}(D_1) \cap \mathcal{H}(D_2) = \emptyset$.

On the basis of Lemmas 9.1–2, the enumerating functions of sets $\mathcal{D}_{\mathrm{I}}$ and $\mathcal{D}_{\mathrm{II}}$ can be evaluated as a function of $\mathcal{D}$'s as

$$f_{\mathcal{D}}(x) = \sum_{D \in \mathcal{D}} x^{m(D)} \tag{9.4}$$

where $m(D)$ is the size of D.

Because $\mathcal{D}_{\mathrm{I}}$ only consists of the trivial map ϑ and ϑ has no edge,

$$f_{\mathcal{D}_{\mathrm{I}}}(x) = 1. \tag{9.5}$$

Lemma 9.3 For $\mathcal{D}_{\mathrm{II}}$,

$$f_{\mathcal{D}_{\mathrm{II}}}(x) = xf_{\mathcal{D}} + 2x^2\frac{\mathrm{d}f_{\mathcal{D}}}{\mathrm{d}x}. \tag{9.6}$$

Proof From Lemma 9.2,

$$\begin{aligned} f_{\mathcal{D}_{\mathrm{II}}}(x) &= \sum_{D\in\mathcal{D}_{\mathrm{II}}} x^{m(D)} \\ &= x\left(\sum_{D\in\mathcal{D}}(2m(D)+1)x^{m(D)}\right) \\ &= x\sum_{D\in\mathcal{D}} x^{m(D)} + 2x\sum_{D\in\mathcal{D}} m(D)x^{m(D)} \\ &= xf_{\mathcal{D}} + 2x^2\frac{\mathrm{d}f_{\mathcal{D}}}{\mathrm{d}x} \end{aligned}$$

where $f_{\mathcal{D}} = f_{\mathcal{D}}(x)$. This is (9.6). □

Theorem 9.1 The differential equation about h

$$\begin{cases} 2x^2\dfrac{\mathrm{d}h}{\mathrm{d}x} = -1 + (1-x)h; \\ h_0 = h|_{x=0} = 1 \end{cases} \tag{9.7}$$

is well defined in the ring of infinite series with integral coefficients and finite terms of negative exponents. And, the solution is $h = f_{\mathcal{D}}(x)$.

Proof Suppose $h = H_0 + H_1x + H_2x^2 + \cdots + H_mx^m + \cdots$, for $H_i \in Z_+,\ i \geq 0$. According to the first relation of (9.7), via equating the coefficients of the terms with the same power of x on its two sides, the recursion

$$\begin{cases} -1 + H_0 = 0; \\ H_1 - H_0 = 0; \\ H_m = (2m-1)H_{m-1},\ m \geq 2 \end{cases} \tag{9.8}$$

is soon found. From this, $H_0 = 1$(the initial condition!), $H_1 = 1, \cdots$, and hence all the coefficients of h can be determined. Because only addition and multiplication are used in the evaluation, all H_m, $m \geq 1$, are integers from integrality of H_0. This is the first statement.

As for the last statement, from (9.1) and (9.5–6), it is seen that $h = f_{\mathcal{D}}(x)$ satisfies the first relation of (9.7). And from the initial condition $h_0 = f_{\mathcal{D}}(0) = 1$, we only have that $h = f_{\mathcal{D}}(x)$ by the first statement. $\square$

In fact, from (9.8),

$$H_m = \prod_{i=1}^{m}(2i-1) = \frac{(2m)!}{2^m m!},$$

where $m \geq 0$.

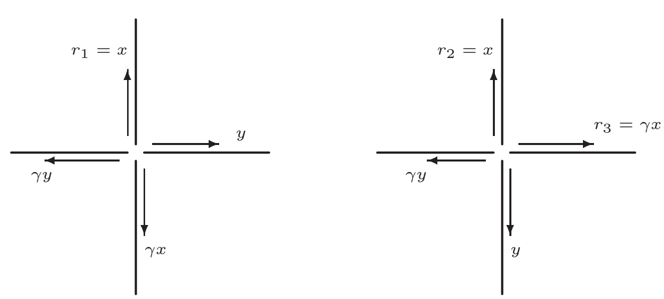
Further, from Theorem 9.2,

$$f_{\mathcal{D}}(x) = 1 + \sum_{m\geq 1} \frac{(2m-1)!}{2^{m-1}(m-1)!} x^m. \tag{9.9}$$

Example 9.1 From (9.9), there are 3 orientable rooted petal bundles of size 2.

However, there are 2 orientable non-rooted petal bundles as shown in (a) and (b) of Fig.9.1.

In (a), based on primal trail code(or dual trail code), only 1 rooted(r_1 as the root) element. In (b), 2 rooted(r_2 and r_3 as the roots) elements.



(a) (b)

Fig.9.1 Petal bundles of size 2

IX.2 Planar pedal bundles

Petal bundles are here restricted to those of genus 0, *i.e.*, *planar pedal bundle*. Rooted is still considered. Because orientable petal bundles can be partitioned into classes by genus as

$$\mathcal{D} = \sum_{k\geq 0} \mathcal{D}_k \tag{9.10}$$

where $\mathcal{D}_k$ is the set of rooted petal bundles with orientable genus k. What is discussed in this section is $\mathcal{D}_0$. For convenience, the trivial map ϑ is included in $\mathcal{D}_0$.

For this, $\mathcal{D}_0$ should be partitioned by the valency of root-face into classes as

$$\mathcal{D}_0 = \sum_{s\geq 0} \mathcal{F}_s \tag{9.11}$$

where $\mathcal{F}_s$, $s \geq 0$, is planar rooted petal bundles with the root-face of valency s.

Lemma 9.4 Let $\mathcal{S}$(the trivial map ϑ is included) and $\mathcal{T}$(ϑ is excluded) be two set of rooted maps. If for any $S = (\mathcal{X}, \mathcal{P}) \in \mathcal{S} - \vartheta$, there exist an integer $k \geq 1$ and maps $S_i = (\mathcal{X}_i, \mathcal{P}_i) \in \mathcal{T}$, $1 \leq i \leq k$, such that

$$\mathcal{X} = \sum_{i=1}^{k} \mathcal{X}_i, \tag{9.12}$$

and $\mathcal{P}$ is different from $\mathcal{P}_i$, $1 \leq i \leq k$, only at vertex

$$(r)_{\mathcal{P}} = (\langle r_1\rangle_{\mathcal{P}_1}, \langle r_2\rangle_{\mathcal{P}_2}, \cdots, \langle r_k\rangle_{\mathcal{P}_k}) \tag{9.13}$$

where $r = r_1$. Then,

$$f_{\mathcal{S}}(x) = \frac{1}{1 - f_{\mathcal{T}}(x)} \tag{9.14}$$

where $f_{\mathcal{S}}(x)$ and $f_{\mathcal{T}}(x)$ are the enumerating functions of, respectively, $\mathcal{S}$ and $\mathcal{T}$.

Proof First, $\mathcal{S}$ is classified based on k mentioned above, $k \geq 0$, *i.e.*,

$$\mathcal{S} = \sum_{k\geq 0} \mathcal{S}_k.$$

Naturally, $\mathcal{S}_0 = \{\vartheta\}$. Then, because any $M_k = (\mathcal{Y}_k, \mathcal{Q}_k) \in \mathcal{S}_k$, $k \geq 1$, has the form as shown in (9.12) and (9.13)($\mathcal{X}$ and $\mathcal{P}$ are, respectively, replaced by $\mathcal{Y}_k$ and $\mathcal{Q}_k$), we have

$$\begin{aligned} f_{\mathcal{S}_k}(x) &= \sum_{M_k \in \mathcal{S}_k} x^{m(M_k)} \\ &= \sum_{\substack{(S_1, S_2, \cdots, S_k) \\ S_i \in \mathcal{T},\ 1 \leq i \leq k}} x^{m(S_1)+m(S_2)+\cdots+m(S_k)} \\ &= \big(f_{\mathcal{T}}(x)\big)^k. \end{aligned}$$

Therefore, by considering $f_{\mathcal{S}_0}(x) = 1$,

$$\begin{aligned} f_{\mathcal{S}}(x) &= \sum_{k \geq 0} f_{\mathcal{S}_k}(x) \\ &= 1 + \sum_{k \geq 1} \big(f_{\mathcal{T}}(x)\big)^k \\ &= \frac{1}{1 - f_{\mathcal{T}}(x)}. \end{aligned}$$

Notice that since x is an undeterminate, it can be considered for the values satisfying $|f_{\mathcal{T}}(x)| < 1$. This lemma is proved. $\square$

If $\mathcal{S}$ and $\mathcal{T}$ are, respectively, seen as $\mathcal{D}_0$ and $\mathcal{F}_1$, it can be checked that the condition of Lemma 9.4 is satisfied, then

$$f_{\mathcal{D}_0}(x) = \frac{1}{1 - f_{\mathcal{F}_1}(x)}. \tag{9.15}$$

Further, another relation between $f_{\mathcal{D}_0}(x)$ and $f_{\mathcal{F}_1}(x)$ has to be found.

Lemma 9.5 Let $\mathcal{F}_{\langle 1 \rangle} = \{D - a | \forall D \in \mathcal{F}_1\}$. Then,

$$\mathcal{F}_{\langle 1 \rangle} = \mathcal{D}_0. \tag{9.16}$$

Proof Because $\vartheta \notin \mathcal{F}_1$, for any $D \in \mathcal{F}_1$, from the planarity of D, $D' = D - a$ is planar and from D with a single vertex, $D' = D - a$ is with a single vertex. Hence, $D' \in \mathcal{D}_0$. This implies that$\mathcal{F}_{\langle 1 \rangle} \subseteq \mathcal{D}_0$.

Conversely, for any $D' = (\mathcal{X}', \mathcal{P}') \in \mathcal{D}_0$, in view of a single vertex, $\mathcal{P}' = (r')_{\mathcal{P}'}$. Let

$$D = D' + a = (\mathcal{X}' + Kr, \mathcal{P})$$

where $\mathcal{P} = (r, \langle r' \rangle_{\mathcal{P}'}, \gamma r)$. Naturally, D is of single vertex. Because D is obtained from D' by appending an edge, from Corollary 4.2 and Lemma 4.6, the planarity of D' leads that D is planar. And, from $(r)_{\mathcal{P}\gamma} = (r)$, $D \in \mathcal{F}_1$. Since $D' = D - a$, $D' \in \mathcal{F}_{\langle 1 \rangle}$. This implies that $\mathcal{D}_0 \subseteq \mathcal{F}_{\langle 1 \rangle}$. □

Because this lemma provides a 1–to–1 correspondence between $\mathcal{F}_1$ and $\mathcal{D}_0$, it is soon obtained that

$$\begin{aligned} f_{\mathcal{F}_1}(x) &= \sum_{D \in \mathcal{F}_1} x^{m(M)} \\ &= x \sum_{D \in \mathcal{D}_0} x^{m(D)} \\ &= x f_{\mathcal{D}_0}(x). \end{aligned} \tag{9.17}$$

In virtue of (9.17) and (9.15),

$$f_{\mathcal{D}_0}(x) = \frac{1}{1 - x f_{\mathcal{D}_0}(x)}. \tag{9.18}$$

Theorem 9.2 Let $h^{(0)} = f_{\mathcal{D}_0}(x)$ be the enumerating function of planar rooted petal bundles with the size as the parameter, then

$$h^{(0)} = \sum_{m \geq 0} \frac{(2m)!}{m!(m+1)!}. \tag{9.19}$$

Proof From (9.18), it is seen that $h^{(0)}$ satisfies the quadratic equation about h as

$$xh^2 - h + 1 = 0.$$

It can be checked that only one of its two solutions is in a power series with all coefficients non-negative integers. That is

$$h^{(0)} = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

By expanding $\sqrt{1-4x}$ into a power series, (9.19) is soon found via rearrangement. $\square$

From the quadratic equation, a non-linear recursion can be derived for determining the coefficients of h. However, a linear recursion can be extracted for getting a simple result. This is far from an universal way.

Example 9.2 From known by (9.19), the number of planar rooted petal bundles of size 3 is 5. However, there are 2 planar nonrooted petal bundles altogether, shown in (a) and (b) of Fig.9.2. In (a), by primal trail codes(or dual trail codes), 3. Their roots are r_1, r_2 and r_3. In (b), 2. Their roots are r_4 and r_5.

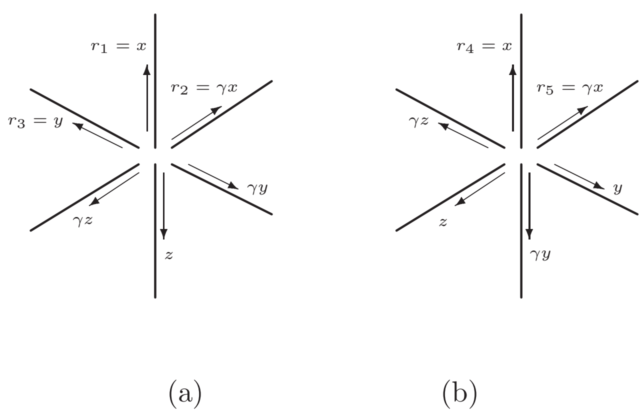


Fig.9.2 Planar petal bundles of size 3

IX.3 Nonorientable pedal bundles

The central task of this section is to determine the enumerating function of nonorientable rooted petal bundles with size as the parameter.

Let $\mathcal{U}$ be the set of all nonorientable rooted petal bundles. Because the trivial map is orientable, ϑ is never in $\mathcal{U}$. In other words, any map in $\mathcal{U}$ does have at least one edge. Now, $\mathcal{U}$ is partitioned into

two classes: $\mathcal{U}_{\mathrm{I}} = \{M|\forall M \in \mathcal{U}, M - a \text{ orientable}\}$ and

$$\mathcal{U}_{\mathrm{II}} = \{M|\forall M \in \mathcal{U}, M - a \text{ nonorientable}\},$$

i.e.,

$$\mathcal{U} = \mathcal{U}_{\mathrm{I}} + \mathcal{U}_{\mathrm{II}}. \tag{9.20}$$

First, the decomposition of the two sets $\mathcal{U}_{\mathrm{I}}$ and $\mathcal{U}_{\mathrm{II}}$ should be investigated.

Lemma 9.6 Let $\mathcal{U}_{\langle \mathrm{I} \rangle} = \{M - a|\forall M \in \mathcal{U}_{\mathrm{I}}\}$. Then,

$$\mathcal{U}_{\langle \mathrm{I} \rangle} = \mathcal{D} \tag{9.21}$$

where $\mathcal{D}$ is the set of all orientable rooted petal bundles given by (9.1).

Proof For $M = (\mathcal{X}, \mathcal{P}) \in \mathcal{U}_{\langle \mathrm{I} \rangle}$, if $M \neq \vartheta$, then $M' = (\mathcal{X}', \mathcal{P}')$ where $\mathcal{X}' = \mathcal{X} + Kr'$ and $\mathcal{P}'$ is different from $\mathcal{P}$ only at the vertex

$$(r')_{\mathcal{P}'} = (r', \beta r', r, \mathcal{P}r, \mathcal{P}^2 r, \cdots, \mathcal{P}^{-1} r)$$

such that $M = M' - a'$. From $M' \in \mathcal{M}_{\mathrm{I}}$, $M \in \mathcal{M}$. If $M = \vartheta$, then

$$M' = (Kr', (r', \beta r')) \in \mathcal{U}_{\mathrm{I}}$$

such that $M = M' - a'$. Meanwhile, $M \in \mathcal{D}$. Hence, $\mathcal{U}_{\langle \mathrm{I} \rangle} \subseteq \mathcal{D}$.

Conversely, for $M = (\mathcal{X}, \mathcal{P}) \in \mathcal{D}$, let $M' = (\mathcal{X}', \mathcal{P}')$ such that $\mathcal{X}' = \mathcal{X} + Kr'$. Because M has a single vertex,

$$\mathcal{P}' = (r')_{\mathcal{P}'} = (r', \beta r', r, \mathcal{P}r, \mathcal{P}^2 r, \cdots, \mathcal{P}^{-1} r).$$

Therefore, M' has a single vertex as well. And, since $r', \beta r' \in \{r'\}_{\Psi'}$, $M' \in \mathcal{U}$. By reminding that $M = M' - a'$ is orientable, $M' \in \mathcal{U}_{\mathrm{I}}$. Thus, $M \in \mathcal{U}_{\langle \mathrm{I} \rangle}$. This implies that $\mathcal{D} \subseteq \mathcal{U}_{\langle \mathrm{I} \rangle}$. □

Lemma 9.7 For any $D = (\mathcal{X}, \mathcal{F}) \in \mathcal{D}$, $r = r(D)$, let $\mathcal{B}(D) = \{B_i|0 \leq i \leq 2m(D)\}$ where $m(D)$ is the size of D, and

$$B_i(D) = \begin{cases} (r', \beta r', \langle r \rangle_{\mathcal{F}}),\ i = 0; \\ (r', r, \cdots, \beta r', \mathcal{F}^i r, \cdots), \\ \qquad 1 \leq i \leq 2m(D) - 1; \\ (r', r, \cdots, \mathcal{F}^i r, \cdots, \beta r'), i = 2m(D). \end{cases} \tag{9.22}$$

Then,

$$\mathcal{U}_{\mathrm{I}} = \sum_{D \in \mathcal{D}} \mathcal{B}(D). \tag{9.23}$$

Proof For any $M = (\mathcal{Z}, \mathcal{P}) \in \mathcal{U}_{\mathrm{I}}$, because

$$D = M - a \in \mathcal{D},$$

M is only some B_i, $1 \leq i \leq 2m(D)$ in (9.22) such that $\mathcal{P}r' = r$, or $\mathcal{P}^2 r' = r$(Here, r' and r are, respectively, the roots of M and D). Therefore, M is also an element of the set on the right hand side of (9.23).

Conversely, for an element M in the set on the right of (9.23), from Lemma 9.6, $M \in \mathcal{U}_{\mathrm{I}}$. This is to say that M is also an element of the set on the left hand side of (9.23). □

Example 9.3 Let $D = (Kx, (x, \gamma x)) \in \mathcal{D}$. Then, D is of size 1, *i.e.*, $m(D) = 1$.

Three rooted petal bundles $B_0(D)$, $B_1(D)$, and $B_2(D) \in \mathcal{U}_{\mathrm{I}}$ are produced from D and shown as, respectively in (a), (b), and (c) of Fig.9.4 where $r' = y$ and $r = x$.

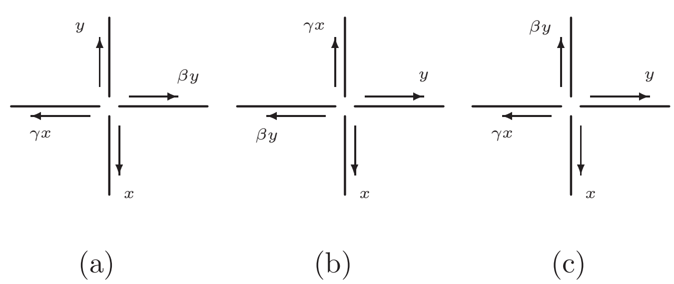


Fig.9.3 Nonorientable petal bundles from orientable ones

Lemma 9.8 Let $\mathcal{U}_{\langle \mathrm{II} \rangle} = \{M - a | \forall M \in \mathcal{U}_{\mathrm{II}}\}$. Then,

$$\mathcal{U}_{\langle \mathrm{II} \rangle} = \mathcal{U}. \tag{9.24}$$

Proof For $M = (\mathcal{X}, \mathcal{P}) \in \mathcal{U}_{\langle \mathrm{II} \rangle}$, let $M' \in \mathcal{U}_{\mathrm{II}}$ such that $M = M' - a'$. Because M' is a nonorientable petal bundle and $M' \in \mathcal{U}_{\mathrm{II}}$, M is a nonorientable petal bundle as well, *i.e.*, $M \in \mathcal{U}$.

Conversely, for any $M = (\mathcal{X}, \mathcal{P}) \in \mathcal{U}$, there exists $M' = (\mathcal{X}', \mathcal{P}') \in \mathcal{U}_{\mathrm{II}}$ such that $M = M' - a'$, *e.g.*, $\mathcal{X}' = \mathcal{X} + Kr'$, $\mathcal{P}' = (r', \gamma r', \langle r\rangle_{\mathcal{P}})$. Therefore, $M \in \mathcal{U}_{\langle \mathrm{II}\rangle}$. □

Further, observe that for a map $M \in \mathcal{U}$, how many non-isomorphic maps $M' \in \mathcal{U}_{\mathrm{II}}$are there such that $M = M' - a'$. Two cases should be considered: (1) r', $\gamma r'$ and r are in the same orbit of $\mathcal{P}'$; (2) r', $\beta r'$ and r are in the same orbit of $\mathcal{P}'$.

(1) Based on the rule of rooting, because $\gamma r'$ only has $2m(M)+1$ possible positions, *i.e.*,

$$\gamma r' = \mathcal{P}'r', \mathcal{P}'r, \mathcal{P}'(\mathcal{P}r), \cdots, \mathcal{P}'(\mathcal{P}^{2m(M)-1}r),$$

then

$$I_i(M) = \begin{cases} (r', \gamma r', \langle r\rangle_{\mathcal{P}}), \ i = 0; \\ (r', r, \cdots, \gamma r', \mathcal{P}^i r, \cdots), \\ \qquad\qquad 1 \le i \le 2m(M) - 1; \\ (r', \langle r\rangle_{\mathcal{P}}, \gamma r'), \ i = 2m(M). \end{cases} \tag{9.25}$$

(2) Based on the rule of rooting, because $\beta r'$ also has $2m(M)+$ 1possible positions, *i.e.*,

$$\beta r' = \mathcal{P}'r', \mathcal{P}'r, \mathcal{P}'(\mathcal{P}r), \cdots, \mathcal{P}'(\mathcal{P}^{2m(M)-1}r),$$

then

$$J_i(M) = \begin{cases} (r', \beta r', \langle r\rangle_{\mathcal{P}}), \ i = 0; \\ (r', r, \cdots, \beta r', \mathcal{P}^i r, \cdots), \\ \qquad\qquad 1 \le i \le 2m(M) - 1; \\ (r', \langle r\rangle_{\mathcal{P}}, \beta r'), \ i = 2m(M). \end{cases} \tag{9.26}$$

Example 9.4 Let $M = (Kx, (x, \beta x)) \in \mathcal{U}$. The map M has only one edge, *i.e.*, $m(M) = 1$.

Six nonorientable petal bundles $I_0(M)$, $I_1(M)$, and $I_2(M)$, with $J_0(M)$, $J_1(M)$, and $J_2(M) \in \mathcal{U}_{\mathrm{II}}$ are produced for M and shown as, respectively, in (a), (b), and (c), with (d), (e), and (f) of Fig.9.4 where $r' = y$ and $r = x$.

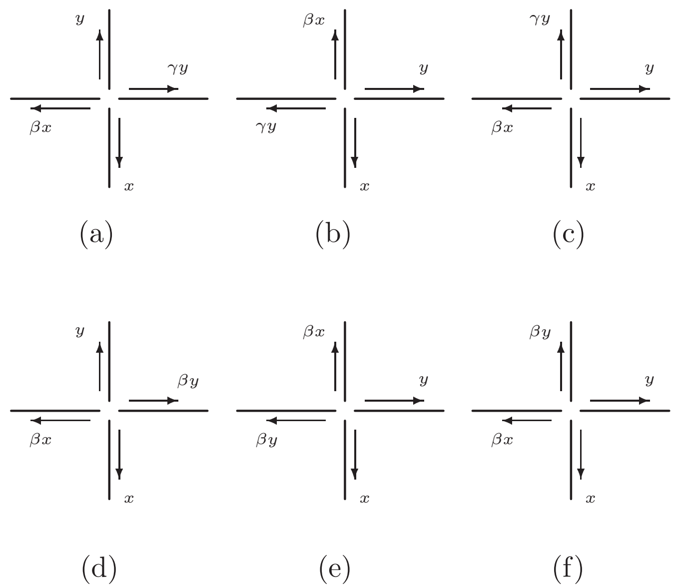


Fig.9.4 Nonorientable petal bundles from nonorientable ones

Lemma 9.9 For any $M \in \mathcal{U}$, let

$$\begin{cases} \mathcal{I}(M) = \{I_i(M)|0 \leq i \leq 2m(M)\}; \\ \mathcal{J}(M) = \{J_j(M)|0 \leq j \leq 2m(M)\}. \end{cases} \tag{9.27}$$

Then,

$$\mathcal{U}_{\mathrm{II}} = \sum_{M \in \mathcal{U}} (\mathcal{I}(M) + \mathcal{J}(M)). \tag{9.28}$$

Proof For any $M' = (\mathcal{X}', \mathcal{P}') \in \mathcal{U}_{\mathrm{II}}$, because $M = M' - a' \in \mathcal{U}$, M' is only some I_i, $0 \leq i \leq 2m(M) - 1$ in (9.25), or some J_j, $0 \leq j \leq 2m(M) - 1$ in (9.26) such that $\mathcal{P}r' = r$, or $\mathcal{P}^2 r' = r$(Here, r' and r are, respectively, the roots of M' and M). Therefore, M is also an element of the set on the right hand side of (9.28).

Conversely, for an element M in the set on the right of (9.28), from Lemma 9.8, $M \in \mathcal{U}_{\mathrm{II}}$. This is to say that M is also an element of the set on the left hand side of (9.28). $\square$

Lemma 9.10 Let $\mathcal{S}$ and $\mathcal{T}$ be two sets of maps. If for any $T \in \mathcal{T}$, there exists a set $\mathcal{L}(T) \subseteq \mathcal{S}$ such that

(i) for any $T \in \mathcal{T}$, $|\mathcal{L}(T)| = am(T) + b$ and for $S \in \mathcal{S}, m(T) = m(S) - c$, where a b and c are constants and $m(T)$ is an isomorphic invariant, *e.g.*, the size; and

(ii) $\mathcal{S} = \sum_{T\in\mathcal{T}} \mathcal{L}(T)$,

then

$$f_{\mathcal{S}}(x) = x^c(bf_{\mathcal{T}} + ax\frac{\mathrm{d}f_{\mathcal{T}}}{\mathrm{d}x}). \tag{9.29}$$

Proof Because $\mathcal{L}(T)$ provides a mapping from a map in $\mathcal{T}$ to a subset of $\mathcal{S}$ and the cardinality of the subset is only dependent on the parameter of the enumerating function(by (i)), and (ii) means that the mapping provides a partition on $\mathcal{S}$, then

$$\begin{aligned} x^{-c}f_{\mathcal{S}}(x) &= \sum_{T\in\mathcal{T}}(am(T) + b)x^{m(T)} \\ &= b\sum_{T\in\mathcal{T}} x^{m(T)} + ax\sum_{T\in\mathcal{T}} m(T)x^{m(T)-1} \\ &= bf_{\mathcal{T}} + ax\frac{\mathrm{d}f_{\mathcal{T}}}{\mathrm{d}x}. \end{aligned}$$

This is (9.29) by multiplying x^c to the two sides. □

Theorem 9.3 The enumerating function $g = f_{\mathcal{U}}(x)$ of nonorientable rooted petal bundles in the set $\mathcal{U}$ with size as the parameter satisfies the equation as

$$\begin{cases} 4x^2\dfrac{\mathrm{d}g}{\mathrm{d}x} = (1-2x)g - x(h + 2x\dfrac{\mathrm{d}h}{\mathrm{d}x}); \\ \dfrac{\mathrm{d}g}{\mathrm{d}x}\Big|_{x=0} = 1, \end{cases} \tag{9.30}$$

where $h = f_{\mathcal{D}}(x)$ is the enumerating function of orientable rooted petal bundles given by (9.9).

Proof Because (9.23) and (9.28) provides the mappings from maps $D \in \mathcal{D}$ and $U \in \mathcal{U}$ to, respectively, a subset of $\mathcal{U}_{\mathrm{I}}$ with $2m(U)+1$ elements and a subset of $\mathcal{U}_{\mathrm{II}}$ with $2(2m(U)+1) = 4m(U)+4$ elements,

where D and U are 1 edge less than their images, from Lemma 9.10,

$$f_{\mathcal{U}_{\mathrm{I}}}(x) = x(h + 2x\frac{\mathrm{d}h}{\mathrm{d}x}) = xh + 2x^2\frac{\mathrm{d}h}{\mathrm{d}x}$$

and

$$f_{\mathcal{U}_{\mathrm{II}}}(x) = x(2g + 4x\frac{\mathrm{d}g}{\mathrm{d}x}) = 2xg + 4x^2\frac{\mathrm{d}g}{\mathrm{d}x}.$$

By (9.20) again,

$$g = xh + 2x^2\frac{\mathrm{d}h}{\mathrm{d}x} + 2xg + 4x^2\frac{\mathrm{d}g}{\mathrm{d}x}.$$

Via rearrangement, (9.30) is soon obtained. □

IX.4 The number of pedal bundles

Because (9.9) provides the number of orientable rooted petal bundles with size m, $m \geq 0$, *i.e.*,

$$H_m = \frac{(2m-1)!}{2^{m-1}(m-1)!}, \tag{9.31}$$

for $m \geq 1$. Of curse, $H_0 = 1$.

Here, the number of nonorientable rooted petal bundles with size m is evaluated only by the equation shown in (9.30). Let G_m be the number of nonorientable rooted petal bundles with size m, $m \geq 1$.

In fact, G_m, $m \geq 1$, are determined by the recursion as

$$\begin{cases} G_m = (4m-2)G_{m-1} + H_m, \ m \geq 2; \\ G_1 = 1. \end{cases} \tag{9.32}$$

The solution of the recursion (9.32) is obtained, *i.e.*,

$$\begin{aligned} G_m = {} & \frac{(2m-1)!}{2^{m-1}(m-1)!} + \prod_{i=2}^{m}(4i-2) \\ & + \sum_{i=2}^{m-1}\frac{(2i-1)!}{2^{i-1}(i-1)!}\prod_{j=i+1}^{m}(4j-2). \end{aligned} \tag{9.33}$$

Example 9.5 When $m = 1$, there are one orientable rooted petal bundle of 1 edge, *i.e.*, $M = (Kx, (x, \gamma x))$ and one nonorientable rooted petal bundle of 1 edge, *i.e.*, $N = (Kx, (x, \beta x)) \in \mathcal{U}$.

By appending an edge Ky on M, 3 non-isomorphic nonorientable rooted petal bundles of 2 edges are produced and shown in (a–c) of Fig.9.3.

By appending an edge Ky on N, 6 non-isomorphic nonorientable rooted petal bundles of 2 edges are produced and shown in (a–f) of Fig.9.4. Then, $G_2 = 9$ which is in agreement with that provided by (9.32), or (9.33).

Now, H_m, G_m, and $H_m^{(0)}$ for $m \le 10$ are listed in Table 9.1.

m	H_m	G_m	$H_m^{(0)}$
1	1	1	1
2	3	9	2
3	15	105	5
4	105	1575	14
5	945	29295	42
6	10395	654885	132
7	135135	17162145	429
8	2027025	516891375	1430
9	34459425	17608766175	4862
10	654729075	669787843725	16796

Table 9.1 Numbers of rooted petal bundles in 10 edges

Lemma 9.11 For an integer $m \ge 1$, the number of non-isomorphic nonorientable rooted petal bundles with size m is

$$G_m = (2^m - 1)H_m \tag{9.34}$$

where H_m is given by (9.31).

Proof By induction. When $m = 1$, from $H_1 = 1$, $G_1 = 1$. (9.34) is true.

Assume G_k satisfies (9.34) for any $1 \le k \le m - 1$, $m \ge 2$. Then,

from (9.32),

$$\begin{aligned} G_m &= (4m-2)G_{m-1} + H_m \\ &= (4m-2)((2^{m-1}-1)H_{m-1}) + H_m. \end{aligned}$$

Since it can, from (9.31), be seen that

$$H_m = (2m-1)H_{m-1},$$

we have

$$\begin{aligned} G_m &= (4m-2)(2^{m-1}-1)\frac{H_m}{2m-1} + H_m \\ &= \left(\frac{4m-2}{2m-1}(2^{m-1}-1)+1\right)H_m \\ &= (2(2^{m-1}-1)+1)H_m \\ &= (2^m-1)H_m. \end{aligned}$$

This is (9.34). □

Theorem 9.4 For an integer $m \geq 1$, the number of non-isomorphic rooted petal bundles with size m is

$$2^m(2m-1)!! \tag{9.35}$$

where

$$(2m-1)!! = \prod_{i=1}^{m}(2i-1). \tag{9.36}$$

Proof Because of (9.34), the number of non-isomorphic petal bundles with m edges is

$$H_m + G_m = 2^m H_m. \tag{9.37}$$

By substituting (9.31) into (9.37), we have

$$\begin{aligned} 2^m H_m &= 2^m \times \frac{(2m-1)!}{2^{m-1}(m-1)!} \\ &= 2 \times \frac{(2m-1)!}{(m-1)!} \\ &= 2^m(2m-1)!!. \end{aligned}$$

This is (9.35). □

The theorem above reminds the number of embeddings of the bouquet of size m

$$2^m(2m-1)!$$

derived from (1.10) as a special case. This is $(m-1)!$ times the number of rooted petal bundles with m edges.

Activities on Chapter IX

IX.5 Observations

A map with only one face is called a *unisheet*.

O9.1 Observe that there is a 1–to–1 correspondence between the set of petal bundles and the set of unisheets.

O9.2 Think that what types of graphs can be as the under graph of a unisheet and what type of graphs are not as the under graph of a unisheet.

O9.3 Is any planar graph a under graph of a planar unisheet?

O9.4 Think about three ways to justify a map which is a planar petal bundle.

O9.5 Discuss how to determine that a petal bundle is on the projective plane.

O9.6 Discuss how to determine that a petal bundle is on the torus.

O9.7 Discuss how to determine that a petal bundle is on the Klein bottle.

O9.8 Discuss how to determine that a unisheet is on the plane.

O9.9 Discuss how to determine that a unisheet is on the projective plane.

O9.10 Discuss how to determine that a unisheet is on the torus.

O9.11 Discuss how to determine that a unisheet is on the Klein

bottle.

IX.6 Exercises

E9.1 Show that for any graph G, there exists a unisheet U such that G is the under graph of U.

E9.2 Prove that the number of rooted unisheet with size m is

$$(2m-1)!! = \prod_{i=1}^{m}(2i-1).$$

E9.3 Prove that a unisheet is planar if, and only if, its under graph is a tree.

E9.4 For an integer $m \geq 1$, determine the number of rooted petal bundles of size m on the torus.

E9.5 For an integer $m \geq 1$, determine the number of rooted petal bundles of size m on the projective plane.

A graph is said to be *unicyclic* if it has only one cycle.

E9.6 Prove that a unisheet is on the projective plane if, and only if, its under graph is unicyclic.

A graph is said to be *eves-cyclic* if it has two fundamental circuits incident.

E9.7 Prove that a unisheet is on the torus if, and only if, its under graph is eves-cyclic.

E9.8 For an integer $m \geq 1$, determine the number of non-isomor- phic rooted petal bundles with size m on the projective plane.

E9.9 For an integer $m \geq 1$, determine the number of non-isomor- phic rooted petal bundles with size m on the torus.

E9.10 For an integer $m \geq 1$, determine the number of non-isomorphic rooted petal bundles with size m on the Klein bottle.

E9.11 For an integer $m \geq 1$, determine the number of non-isomorphic rooted unisheets with size m on the projective plane.

E9.12 For an integer $m \geq 1$, determine the number of non-isomorphic rooted unisheets with size m on the Klein bottle.

A map of order 3 is also called *tri-pole map*.

E9.13 For a given integer $m \geq 1$, determine the number of all non-isomorphic rooted tri-pole maps with size m in the plane.

IX.7 Researches

R9.1 For two integers $m \geq 1$ and $p \geq 2$, determine the number of rooted petal bundles with size m on the surface of orientable genus p.

R9.2 For two integers $m \geq 1$ and $q \geq 3$, determine the number of rooted petal bundles with size m on the surface of nonorientable genus q.

R9.3 For two integers $m \geq 1$ and $p \geq 2$, determine the number of rooted unisheets with size m on the surface of orientable genus p.

R9.4 For two integers $m \geq 1$ and $q \geq 3$, determine the number of rooted unisheets with size m on the surface of nonorientable genus q.

A map of order 2 is also called *bi-pole map*.

R9.5 For a given integer $m \geq 1$, determine the number of all non-isomorphic orientable rooted bi-pole maps with size m

R9.6 For a given integer $m \geq 1$, determine the number of all non-isomorphic nonorientable rooted bi-pole maps with size m

R9.7 For two integers $m \geq 1$ and $p \geq 0$, determine the number of all non-isomorphic orientable rooted bi-pole maps with size m of

the surface of genus p.

R9.8 For two integers $m \geq 1$ and $q \geq 1$, determine the number of all non-isomorphic nonorientable rooted bi-pole maps with size m on the surface of genus q.

R9.9 For a given integer $m \geq 1$, determine the number of all non-isomorphic orientable rooted tri-pole maps with size m

R9.10 For a given integer $m \geq 1$, determine the number of all non-isomorphic nonorientable rooted tri-pole maps with size m

R9.11 For two integers $m \geq 1$ and $p \geq 1$, determine the number of all non-isomorphic orientable rooted tri-pole maps with size m of the surface of genus p.

R9.12 For two integers $m \geq 1$ and $q \geq 1$, determine the number of all non-isomorphic nonorientable rooted tri-pole maps with size m on the surface of genus q.

R9.13 For two integers $n \geq 5$ and $p \geq s(n)$ where

$$s(n) = \lceil \frac{(n-3)(n-4)}{12} \rceil,$$

i.e., the least integer not less than the fractional $(n-3)(n-4)/12$(or called the *up-integer* of the fractional), determine the number of rooted maps whose under graph is the complete graph of order n with orientable genus p.

R9.14 For two integers $n \geq 5$ and $p \geq t(n)$ where

$$t(n) = \lceil \frac{(n-3)(n-4)}{6} \rceil,$$

determine the number of rooted maps whose under graph is the complete graph of order n with nonorientable genus q.

R9.15 For two integers $n \geq 3$ and $p \geq c(n)$ where

$$c(n) = (n-4)2^{n-3} + 1,$$

determine the number of rooted maps whose under graph is the n-cube with orientable genus p.

R9.16 For two integers $n \geq 3$ and $q \geq d(n)$ where

$$d(n) = (n-4)2^{n-2} + 2,$$

determine the number of rooted maps whose under graph is the n-cube with nonorientable genus q.

R9.17 For three integers $m, n \geq 3$ and $p \geq r(n)$ where

$$r(n) = \lceil \frac{(m-2)(n-2)}{4} \rceil,$$

determine the number of rooted maps whose under graph is the complete bipartite graph of order $m+n$ with orientable genus p.

R9.18 For three integers $m, n \geq 3$ and $q \geq l(n)$ where

$$l(n) = \lceil \frac{(m-2)(n-2)}{2} \rceil,$$

determine the number of rooted maps whose under graph is the complete bipartite graph of order $m+n$ with nonorientable genus q.

Chapter X

Asymmetrized Maps

- From decomposing the set of rooted orientable maps, a quadratic differential equation satisfied by the enumerating function with size as the parameter is discovered and then a recursion formula is extracted for determining the function.

- A quadratic equation of the enumerating function in company with its partial values for rooted maps on the surface of orientable genus 0 is discovered with an extra parameter and then an explicit expression of the function with only size as a parameter is via characteristic parameters extracted for each term summation free.

- From decomposing the set of rooted nonorientable maps, a nonlinear differential equation satisfied by the enumerating function with size as the parameter is discovered in company with the orientable case and then a recursion formula is extracted for determining the function.

- The numbers of orientable, nonorientable and total maps with given size are, in all, obtained and then calculated for size not greater than 10.

X.1 Orientable equation

It is from Corollary 8.7 shown that a map with symmetry be-

comes a map without symmetry whenever an element is chosen as the root. Such a map with a root is called a *rooted map*.

Rooting is, in fact, a kind of simplification in mathematics, particularly in recognizing distinct combinatorial configurations for reducing the complexity.

As soon as the rooted case is done, the general case can be recovered by considering the symmetry in a suitable way.

For maps, the estimation of the order of the automorphism group of a map in the last chapter and the efficient algorithm for justifying and recognizing if two maps are isomorphic in Chapter VII provide a theoretical foundation for transforming rooted maps into non-rooted maps. This will be seen in the next chapter.

The main purpose of this chapter is to present some methods for investigating non-planar rooted maps as appendix to the monograph Enumerative Theory of Maps[Liu7] in which most pages are for planar maps, particularly rooted.

Let $\mathcal{M}$ be the set of all orientable rooted maps. For $M = (\mathcal{X}, \mathcal{P}) \in \mathcal{M}$, let

$$v_x = (x)_{\mathcal{P}} = (x, \mathcal{P}x, \cdots, \mathcal{P}^{-1}x) \tag{10.1}$$

be the vertex incident with $x \in \mathcal{X}$. The root is always denoted by r. The rooted edge which is incident with r is denoted by $a = Kr$. The rooting of $M - a$ is taking $\mathcal{P}^{\delta}r$ as its root where

$$\delta = \min\{i|\mathcal{P}^i r \notin Kr, i \geq 1\}. \tag{10.2}$$

In fact,

$$\delta = \begin{cases} 1, & \text{if } \mathcal{P}r \neq \gamma r; \\ 2, & \text{othewise.} \end{cases} \tag{10.3}$$

In virtue of Theorem 3.4, $M - a$ is a map if, and only if, a is not a harmonic loop except terminal link (or segmentation edge) of M.

Now, let us partition $\mathcal{M}$ into three parts: $\mathcal{M}_{\mathrm{I}}$, $\mathcal{M}_{\mathrm{II}}$ and $\mathcal{M}_{\mathrm{III}}$, *i.e.*,

$$\mathcal{M} = \mathcal{M}_{\mathrm{I}} + \mathcal{M}_{\mathrm{II}} + \mathcal{M}_{\mathrm{III}} \tag{10.4}$$

where $\mathcal{M}_{\mathrm{I}} = \{\vartheta\}$, *i.e.*, consisted of the trivial map, $\mathcal{M}_{\mathrm{II}}$ and $\mathcal{M}_{\mathrm{III}}$ are, respectively, consisted of those with a as a segmentation edge and not.

Lemma 10.1 Let $\mathcal{M}_{\langle\mathrm{II}\rangle} = \{M - a | \forall M \in \mathcal{M}_{\mathrm{II}}\}$. Then,

$$\mathcal{M}_{\langle\mathrm{II}\rangle} = \mathcal{M} \times \mathcal{M} \tag{10.5}$$

where $\times$ stands for the Cartesian product of two sets.

Proof For any $M = (\mathcal{X}, \mathcal{P}) \in \mathcal{M} \times \mathcal{M}$, let $M = M_1 + M_2$, $M_i = (\mathcal{X}_i, \mathcal{P}_i)$, $i = 1, 2$. Assume $M' = (\mathcal{X}', \mathcal{P}')$ such that $\mathcal{X}' = \mathcal{X} + Kr'$ and $\mathcal{P}'$ is different from $\mathcal{P}_2$ or $\mathcal{P}_1$ only at, respectively,

$$v_{r'} = (r')_{\mathcal{P}'} = (r', r_2, \mathcal{P}_2 r_2, \cdots, \mathcal{P}_2^{-1} r_2)$$

or

$$v_{\beta r'} = (\beta r')_{\mathcal{P}'} = (\alpha\beta r', r_1, \mathcal{P}_1 r_1, \cdots, \mathcal{P}_1^{-1} r_1).$$

Since $M' \in \mathcal{M}$ and its rooted edge $a' = Kr'$ is a segmentation edge, $M' \in \mathcal{M}_{\mathrm{II}}$. It is checked that $M = M' - a'$. Therefore, $M \in \mathcal{M}_{\langle\mathrm{II}\rangle}$.

Conversely, for any $M \in \mathcal{M}_{\langle\mathrm{II}\rangle}$, we have $M' \in \mathcal{M}_{\mathrm{II}}$ such that $M = M' - a'$ where $a' = Kr'$. From $M' \in \mathcal{M}_{\mathrm{II}}$, $M = M_1 + M_2$ where $M_1, M_2 \in \mathcal{M}$. This implies that $M \in \mathcal{M} \times \mathcal{M}$. $\square$

It is seen from this lemma that there is a 1–to–1 correspondence between $M(\in \mathcal{M}_{\langle\mathrm{II}\rangle}$, or $\mathcal{M} \times \mathcal{M})$ and $M'(\in \mathcal{M}_{\mathrm{II}})$. Hence,

$$|\mathcal{M}_{\mathrm{II}}| = |\mathcal{M} \times \mathcal{M}|. \tag{10.6}$$

For $M = (\mathcal{X}, \mathcal{P}) \in \mathcal{M}_{\mathrm{III}}$, because $M - a$ is a map (Theorem 3.4), from (10.3), the root $r(M-a)$ of $M - a$ has two possibilities: when $\mathcal{P}r(M) \neq \gamma r(M)$, $r(M-a) = \mathcal{P}r(M)$; otherwise,

$$r(M-a) = \mathcal{P}^2 r(M).$$

Let $\tilde{M} = (\tilde{\mathcal{X}}, \tilde{\mathcal{P}}) = M - a$ where $\tilde{\mathcal{X}} = \mathcal{X} - Kr$ and $\tilde{\mathcal{P}}$ are different from $\mathcal{P}$ only at

$$\left.\begin{aligned} (\tilde{r})_{\tilde{\mathcal{P}}} &= (\mathcal{P}r, \mathcal{P}^2 r, \cdots, \mathcal{P}^{-1} r), \\ (\mathcal{P}\gamma r)_{\tilde{\mathcal{P}}} &= (\mathcal{P}\gamma r, \mathcal{P}^2\gamma r, \cdots, \mathcal{P}^{-1}\gamma r) \end{aligned}\right\} \text{ if } a \text{ is not a loop;}$$

otherwise, *i.e.*, when a is a loop,

$$(\tilde{r})_{\tilde{\mathcal{P}}} = \begin{cases} (\mathcal{P}^2 r, \mathcal{P}^3 r, \cdots, \mathcal{P}^{-1} r), \text{ 当 } \gamma r = \mathcal{P} r; \\ (\mathcal{P} r, \cdots, \mathcal{P}^{s-1} r, \mathcal{P}^{s+1} r, \cdots, \mathcal{P}^{-1} r), \\ \qquad\qquad \text{if } \gamma r = \mathcal{P}^s r,\ s \geq 2. \end{cases}$$

Since $M(\mathcal{X}, \mathcal{P})$ is orientable, group $\Psi = \Psi_{\{\gamma,\mathcal{P}\}}$ has two orbits $\{r\}_\Psi$ and $\{\alpha r\}_\Psi$ on $\mathcal{X}$. For $\tilde{M} = (\tilde{\mathcal{X}}, \tilde{\mathcal{P}}) = M - a$, group $\tilde{\Psi} = \Psi_{\{\gamma,\tilde{\mathcal{P}}\}}$ also has two orbits

$$\{\tilde{r}\}_{\tilde{\Psi}} = \{r\}_\Psi - \{r, \gamma r\}$$

and

$$\{\alpha\tilde{r}\}_{\tilde{\Psi}} = \{\alpha r\}_\Psi - \{\alpha r, \beta r\}.$$

So, $\tilde{M}$ is also orientable.

Furthermore, for every element $y \in \{\tilde{r}\}_{\tilde{\Psi}}$, is there exactly one position of $a = Kr$, *i.e.*, γr is in the angle $\langle \alpha y, \tilde{\mathcal{P}} \rangle$, for $M \in \mathcal{M}_{\text{III}}$ such that $\tilde{M} = M - a$. This means that in $\mathcal{M}_{\text{III}}$, there are

$$|\{\tilde{r}\}_{\tilde{\Psi}}| = \frac{1}{2}|\tilde{\mathcal{X}}| = 2m(\tilde{M}),$$

where $m(\tilde{M})$ is the size of $\tilde{M}$, non-isomorphic maps for producing $\tilde{M}$. By considering for the case $\mathcal{P} r = \gamma r$, in $\mathcal{M}_{\text{III}}$, there are $2m(\tilde{M}) + 1$ non-isomorphic maps for $\tilde{M}$ altogether.

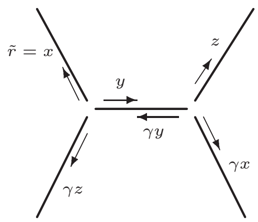
Example 10.1 Let

$$\tilde{M} = (Kx + Ky + Kz, (x, y, \gamma z)(z, \gamma x, \gamma y))$$

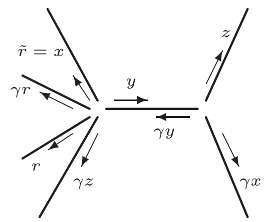
where $\tilde{r} = x$ is the root shown in Fig.10.1(a). Since it is orientable, the orbit of group $\tilde{\Psi}$ which $\tilde{r}$ is in can be written as a cyclic permutation as

$$(\tilde{r})_{\tilde{\Psi}} = (x, y, \gamma z, z, \gamma x, \gamma y).$$

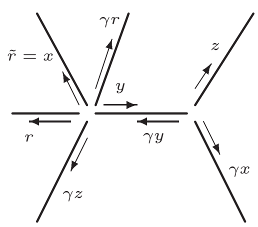
Then, Fig.10.1(b–h) presents all the $2m(\tilde{M}) + 1 = 2 \times 3 + 1 = 7$ maps in $\mathcal{M}_{\text{III}}$, obtained by appending $a = Kr$ on map $\tilde{M}$ where (b) is for $\mathcal{P} r = \gamma r$ and (c–h) are for those obtained by appending $a = Kr$ in the order of $(\tilde{r})_{\tilde{\Psi}}$ from $\tilde{M}$.


$\tilde{r} = x$
y
γy
z
γx
γz

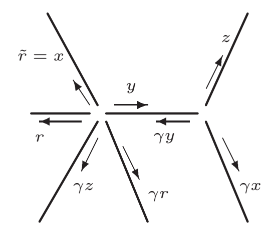
(a)


$\tilde{r} = x$
γr
r
y
γy
z
γx
γz

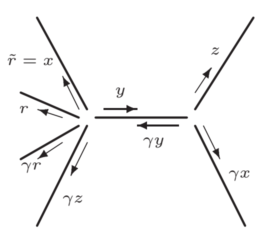
(b)


$\tilde{r} = x$
γr
r
y
γy
z
γx
γz

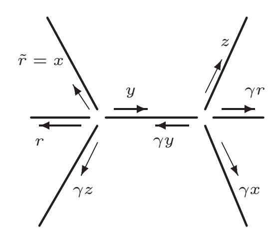
(c)


$\tilde{r} = x$
r
y
γy
z
γx
γz
γr

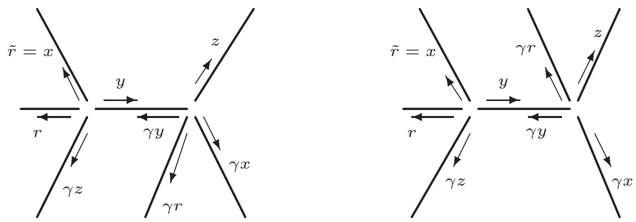
(d)


$\tilde{r} = x$
r
γr
y
γy
z
γx
γz

(e)


$\tilde{r} = x$
r
y
γy
z
γr
γx
γz

(f)



(g) (h)

Fig.10.1 New maps obtained by appending an edge

Lemma 10.2 Let $\mathcal{M}_{\langle\mathrm{III}\rangle} = \{M - a | \forall M \in \mathcal{M}_{\mathrm{III}}\}$. Then,

$$\mathcal{M}_{\langle\mathrm{III}\rangle} = \mathcal{M}. \tag{10.7}$$

Proof Because for any $M \in \mathcal{M}_{\mathrm{III}}$, $M - a$ is also a map (Theorem 3.4), then $\mathcal{M}_{\langle\mathrm{III}\rangle} \subseteq \mathcal{M}$.

Conversely, for any $M \in \mathcal{M}$, any one, *e.g.*, M' of the $2m(M)+1$ maps obtained by appending a' from M in the above way is with $M' \in \mathcal{M}_{\mathrm{III}}$. Because $M = M' - a'$, then $M \in \mathcal{M}_{\langle\mathrm{III}\rangle}$. □

For convenience, let $\mathcal{H}(M)$ be the set of all the $2m(\tilde{M})+1$ maps in $\mathcal{M}_{\mathrm{III}}$, obtained from M by appending an edge in the above way. From Theorem 8.1, they are all mutually nonisomorphic in the sense of rooting.

Lemma 10.3 For $\mathcal{M}_{\mathrm{III}}$,

$$\mathcal{M}_{\mathrm{III}} = \sum_{M \in \mathcal{M}} \mathcal{H}(M). \tag{10.8}$$

Proof For any $M = (\mathcal{X}, \mathcal{P}) \in \mathcal{M}_{\mathrm{III}}$, let $\tilde{M} = (\tilde{\mathcal{X}}, \tilde{\mathcal{P}}) = M - a$. Because a is not a segmentation edge, from Theorem 3.4 and Corollary 3.1, $\tilde{M} \in \mathcal{M}$. By orientability, because $r \in \{\tilde{r}\}_{\tilde{\Psi}}$ where $\tilde{\Psi} = \Psi_{\{\gamma,\tilde{\mathcal{P}}\}}$, there exists $y \in \{\tilde{r}\}_{\tilde{\Psi}}$ such that $\mathcal{P}y = \gamma r$, or $\mathcal{P}r = \gamma r$. Because $|\{\tilde{r}\}_{\tilde{\Psi}}| = 2m\tilde{M}$, the former has $2m(\tilde{M})$ possibilities and the latter, only one. This is the $2m(\tilde{M})+1$ possibilities in $\mathcal{H}(\tilde{M})$. Further,

because $\tilde{M} \in \mathcal{M}$, M is an element of the set on the right hand side of (10.8).

Conversely, for any $M \in \mathcal{H}(\tilde{M})$, $\tilde{M} \in \mathcal{M}$, Since $\tilde{M} = M - a \in \mathcal{M}$, by Theorem 3.4 and Corollary 3.1, a is not a segmentation edge. Therefore, $M \in \mathcal{M}_{\mathrm{III}}$. □

Furthermore, (10.8) provides a 1–to–1 correspondence between the sets on its two sides. This enables us to construct all orientable maps with the rooted edge not a segmentation edge from general orientable maps with smaller size.

In order to determine the number of non-isomorphic orientable rooted maps in $\mathcal{M}$ with size $m \geq 0$, the *enumerating function* of set $\mathcal{M}$

$$f_{\mathcal{M}}(x) = \sum_{M \in \mathcal{M}} x^{m(M)} \tag{10.9}$$

has to be investigated for a simpler form in infinite power series where $m(M)$ is the size of M. In the series form of (10.9), the coefficient of the term with x^m, $m \geq 0$, is just the number of non-isomorphic orientable rooted maps with size m.

From (10.4),

$$f_{\mathcal{M}}(x) = f_{\mathcal{M}_{\mathrm{I}}}(x) + f_{\mathcal{M}_{\mathrm{II}}}(x) + f_{\mathcal{M}_{\mathrm{III}}}(x). \tag{10.10}$$

Lemmas above enable us to evaluate $f_{\mathcal{M}_{\mathrm{I}}}(x)$, $f_{\mathcal{M}_{\mathrm{II}}}(x)$ and $f_{\mathcal{M}_{\mathrm{III}}}(x)$ as functions of $f = f_{\mathcal{M}}(x)$.

First, because $\mathcal{M}_{\mathrm{I}}$ contains only one map ϑ and $m(\vartheta) = 0$, $f_{\mathcal{M}_{\mathrm{I}}}(x)$ contributes the constant term 1 of f, *i.e.*,

$$f_{\mathcal{M}_{\mathrm{I}}}(x) = 1. \tag{10.11}$$

Lemma 10.4 For $\mathcal{M}_{\mathrm{II}}$,

$$f_{\mathcal{M}_{\mathrm{II}}}(x) = xf^2. \tag{10.12}$$

Proof According to the 1–to–1 correspondence between $\mathcal{M}_{\mathrm{II}}$ and $\mathcal{M}_{\langle\mathrm{II}\rangle}$ and that the former is with its size 1 greater than the latter

in the correspondence, by (10.6),

$$\begin{aligned} f_{\mathcal{M}_{\mathrm{II}}}(x) &= x \sum_{M \in \mathcal{M}_{\langle \mathrm{II} \rangle}} x^{m(M)} \\ &= x \sum_{M \in \mathcal{M} \times \mathcal{M}} x^{m(M)} \\ &= x (\sum_{M \in \mathcal{M}} x^{m(M)})^2 \\ &= x f^2. \end{aligned}$$

This is (10.12). □

Lemma 10.5 For $\mathcal{M}_{\mathrm{III}}$,

$$f_{\mathcal{M}_{\mathrm{III}}}(x) = xf + 2x^2 \frac{\mathrm{d}f}{\mathrm{d}x}. \tag{10.13}$$

Proof From the 1–to–1 correspondence between $\mathcal{M}_{\mathrm{II}}$ and $\mathcal{M}_{\langle \mathrm{II} \rangle}$ and that the former is with its size 1 greater than the latter in the correspondence, and then by Lemma 10.3 and Lemma 9.10,

$$\begin{aligned} f_{\mathcal{M}_{\mathrm{III}}}(x) &= x \sum_{M \in \mathcal{M}_{\langle \mathrm{III} \rangle}} x^{m(M)} \\ &= x(f + 2x \frac{\mathrm{d}f}{\mathrm{d}x}) \\ &= xf + 2x^2 \frac{\mathrm{d}f}{\mathrm{d}x}. \end{aligned}$$

This is (10.13). □

Theorem 10.1 The differential equation about f

$$\begin{cases} 2x^2 \dfrac{\mathrm{d}f}{\mathrm{d}x} = -1 + (1-x)f - xf^2; \\ f_0 = f|_{x=0} = 1 \end{cases} \tag{10.14}$$

is well defined in the ring of infinite power series with all coefficients nonnegative integers and the terms of negative powers finite. And, the solution is $f = f_{\mathcal{M}}(x)$.

Proof Suppose $f = F_0 + F_1x + F_2x^2 + \cdots + F_mx^m + \cdots$, $F_i \in Z_+$, $i \geq 0$. Based on the first relation of (10.14), by equating the coefficients on the two sits with the same power of x, the recursion

$$\begin{cases} -1 + F_0 = 0; \\ F_1 - F_0 - F_0^2 = 0; \\ F_m = (2m-1)F_{m-1} + \sum_{i=0}^{m-1} F_iF_{m-1-i}, \\ \qquad m \geq 2 \end{cases} \tag{10.15}$$

is soon extracted. Then, $F_0 = 1$(the initial condition), $F_1 = 2, \cdots$, all the coefficients of f can uniquely found from this recursion. Because only addition and multiplication are used for evaluating all the coefficients from the initial condition that F_0 is an integer, F_m, $m \geq 1$, must all be integers. This is the first statement.

For the last statement, from (10.10) and (10.11–13), it is seen that $f = f_{\mathcal{M}}(x)$ satisfies the first relation of (10.14). And, $f_0 = f_{\mathcal{M}}(0) = 1$ is just the initial condition. By the uniqueness in the first statement, the only possibility is $f = f_{\mathcal{M}}(x)$. □

Although the form of the equation in Theorem 10.1 is rather simple, because of the occurrence of f^2, it is far from getting the solution directly. In fact, it is an equation in the Riccati's type. It has no analytic solution in general.

X.2 Planar rooted maps

Let $\mathcal{T}$ be the set of all planar rooted maps. Because it looks hard to decompose $\mathcal{T}$ into some classes so that each class can be produced by $\mathcal{T}$ with only size as the parameter. Now, another parameter for a map M, *i.e.*, the valency of the rooted vertex $n(M)$, is introduced. The enumerating function of $\mathcal{T}$ is

$$t(x, y) = f_{\mathcal{T}}(x, y) = \sum_{M \in \mathcal{T}} x^{m(M)} y^{n(M)} \tag{10.16}$$

where $m(M)$ is still the size of M.

Assume that $\mathcal{T}$ is partitioned into three classes: $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$, *i.e.*,

$$\mathcal{T} = \mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2 \tag{10.17}$$

where $\mathcal{T}_0 = \{\vartheta\}$, $\mathcal{T}_1$ and $\mathcal{T}_1$ are the sets of planar rooted maps with the rooted edge, respective a loop and a link(not loop).

For $M \in \mathcal{T}$, let $a = Kr$ be the rooted edge of M with the root $r = r(M)$. For maps $M_i \in \mathcal{T}$(i=1 and 2), let $a_i = Kr_i$ be the rooted edge of M_i with the root $r_i = r(M_i)$.

The 1-*addition* of two maps $M_1 = (\mathcal{X}_1, \mathcal{P}_1)$ and $M_2 = (\mathcal{X}_2, \mathcal{P}_2)$ is to produce the map $M_1 +\cdot\ M_2 = M_1 \cup M_2$ with the root $r = r_1$ provided $M_1 \cap M_2 = \{v_r\}$ where $v_r = (\langle r_1\rangle_{\mathcal{P}_1}, \langle r_2\rangle_{\mathcal{P}_2})$.

Lemma 10.6 Let $\mathcal{T}_{\langle 1\rangle} = \{M - a | \forall M \in \mathcal{T}_1\}$, then

$$\mathcal{T}_{\langle 1\rangle} = \mathcal{T} \times\cdot\ \mathcal{T} \tag{10.18}$$

where $\mathcal{T} \times\cdot\ \mathcal{T} = \{M_1 + M_2 | \forall M_1, M_2 \in \mathcal{T}\}$ is called the 1-*product* of $\mathcal{T}$ with itself.

Proof For any $M = (\mathcal{X}, \mathcal{P}) \in \mathcal{T}_{\langle 1\rangle}$, let $M' = (\mathcal{X}', \mathcal{P}') \in \mathcal{T}_1$ such that $M' - a' = M$. Because $a' = Kr'$ is a loop,

$$(r')_{\mathcal{P}'} = (r', \mathcal{P}'r', \cdots, \gamma r', \cdots, \mathcal{P}'^{-1}r').$$

From the planarity, $M' - a' = M_1 +\cdot\ M_2$, $M_i = (\mathcal{X}_i, \mathcal{P}_i)$, $i = 1, 2$, where $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2 = \mathcal{X}' - Kr'$, $\mathcal{P}_1$ and $\mathcal{P}_2$ are different from $\mathcal{P}$ only at $(r)_{\mathcal{P}}$ becoming, respectively,

$$(r_1)_{\mathcal{P}_1} = (\mathcal{P}'r', \mathcal{P}(\mathcal{P}'r'), \cdots, \mathcal{P}'^{-1}\gamma r')$$

where $\gamma = \alpha\beta$ and

$$(r_2)_{\mathcal{P}_2} = (\mathcal{P}'\gamma r', \mathcal{P}(\mathcal{P}'\gamma r'), \cdots, \mathcal{P}'^{-1}r').$$

This implies $M \in \mathcal{T} \times\cdot\ \mathcal{T}$.

Conversely, for $M \in \mathcal{T} \times\cdot\ \mathcal{T}$, because $M = M_1 + M_2$, let $M' = M + a'$, $a' = Kr'$, such that

$$(r')_{\mathcal{P}'} = (r', \langle r_1\rangle_{\mathcal{P}_1}, \gamma r', \langle r_2\rangle_{\mathcal{P}_2}),$$

then $M' \in \mathcal{T}$ and $M = M' - a'$. Since a' is a loop, $M' \in \mathcal{T}_1$ and hence $M \in \mathcal{T}_{\langle 1 \rangle}$. $\square$

Because this lemma presents a 1–to–1 correspondence $M = M_1 + M_2$ between $M \in \mathcal{T}_{\langle 1 \rangle}$ and $M_1, M_2 \in \mathcal{T}$ with $m(M) = m(M_1) + m(M_2)$ and $n(M) = n(M_1) + n(M_2)$, the enumerating function of $\mathcal{T}_{\langle 1 \rangle}$

$$\begin{aligned} f_{\mathcal{T}_{\langle 1 \rangle}}(x,y) &= \sum_{M \in \mathcal{T} \rtimes \mathcal{T}} x^{m(M)} y^{n(M)} \\ &= \sum_{M_1, M_2 \in \mathcal{T}} x^{m(M_1)+m(M_2)} y^{n(M_1)+n(M_2)} \\ &= (\sum_{M \in \mathcal{T}} x^{m(M)} y^{n(M)})^2 \\ &= t^2(x,y). \end{aligned} \tag{10.19}$$

Then, from the 1–to–1 correspondence between $\mathcal{T}_1$ and $\mathcal{T}_{\langle 1 \rangle}$ with the former of size 1 greater than the latter and the former of the rooted vertex valency 2 greater than the latter, the enumerating function of $\mathcal{T}_1$ is

$$f_{\mathcal{T}_1}(x,y) = xy^2 f_{\mathcal{T}_{\langle 1 \rangle}}(x,y) = xy^2 t^2(x,y). \tag{10.20}$$

However, for $\mathcal{T}_2$, the correspondence between $\mathcal{T}_2$ and $\mathcal{T}_{\langle 2 \rangle} = \{M \bullet a | \forall M \in \mathcal{T}_2\}$ with the former of size 1 greater than the latter is not of 1–to–1 where $M \bullet a$ is the contraction of the rooted edge a on map M. The root on $M \bullet a$ is defined to be $\mathcal{P}\gamma r$ when $\mathcal{P}\gamma r \neq \gamma r$; or $(\mathcal{P}\gamma)^2 r$ otherwise. Because a is a link, this is a basic transformation. According to Chapter V, $M - a$ is planar if, and only if M is. Hence,

$$\mathcal{T}_{\langle 2 \rangle} = \mathcal{T}. \tag{10.21}$$

Further, observe what a correspondence between $\mathcal{T}_2$ and $\mathcal{T}$ is.

For $M = (\mathcal{X}, \mathcal{P}) \in \mathcal{T}$, let $(r)_{\mathcal{P}} = (r, \mathcal{P}r, \cdots, \mathcal{P}^{n(M)-1}r)$ where $n(M)$ is the valency of the rooted vertex v_r on M. By splitting a link at v_r, all those obtained are still planar because this operation is a basic transformation. For doing this, there are $n(M) + 1$ possibilities altogether.

Let $M_i = (\mathcal{X}_i, \mathcal{P}_i), 0 \leq i \leq n(M)$ be all the $n(M) + 1$ maps obtained from M by splitting an edge at the rooted vertex $v_r = (r)_\mathcal{P}$ as

$$v_{r_i} = \begin{cases} (r_0, \langle r\rangle_\mathcal{P}),\ \text{while } v_{\beta r_0} = (\gamma r_0); \\ (r_1, \mathcal{P}r, \cdots, \mathcal{P}^{n(M)-1}r), \text{while } v_{\beta r_1} = (\gamma r_1, r); \\ (r_2, \mathcal{P}^2 r, \cdots, \mathcal{P}^{n(M)-1}r), \text{while } v_{\beta r_2} = (\gamma r_2, r, \mathcal{P}r); \\ \cdots\cdots \\ (r_{n(M)-1}, \mathcal{P}^{n(M)-1}r), \\ \quad \text{while } v_{\beta r_{n(M)-1}} = (\gamma r_{n(M)-1}, r, \cdots, \mathcal{P}^{n(M)-2}r); \\ (r_{n(M)}), \text{while } v_{\beta r_{n(M)}} = (\gamma r_{n(M)}, \langle r\rangle_\mathcal{P}). \end{cases} \tag{10.22}$$

Lemma 10.7 For a map $M \in \mathcal{T}$, let

$$\mathcal{K}(M) = \{M_i | i = 0, 1, 2, \cdots, n(M)\}$$

where M_i, $0 \leq i \leq n(M)$, are given by (10.22). Then,

$$\mathcal{T}_2 = \sum_{M \in \mathcal{T}} \mathcal{K}(M). \tag{10.23}$$

Proof For any $M \in \mathcal{T}_2$, because $a = Kr$ is a link, from (10.21), $M \bullet a \in \mathcal{T}$ and from (10.22), $M \in \mathcal{K}(M \bullet a)$. Therefore M is an element of the set on the right hand side of (10.23).

Conversely, for M is an element of the set on the right of (10.23), there exists a map $M' \in \mathcal{T}$ such that $M \in \mathcal{K}(M')$. Because all maps in $\mathcal{K}(M)$ are planar if, and only if, M' is planar, $M \in \mathcal{T}$ as well. Moreover, from (10.22), the rooted edge a is a link, $M \in \mathcal{T}_2$. This meas that M an element of the set on the left hand side of (10.23).$\square$

Because (10.23) presents a 1–to–1 correspondence between maps with the same size on its two sides and the valency of rooted vertex of $M_i (0 \leq i \leq n(M))$ is $n(M) - i$ for any $M \in \mathcal{T}$, the enumerating

function of $\mathcal{T}_2$ is

$$\begin{aligned} f_{\mathcal{T}_2}(x,y) &= \sum_{M\in\mathcal{T}_2} x^{m(M)}y^{n(M)} \\ &= xy\sum_{M\in\mathcal{T}}(\sum_{i=0}^{n(M)} y^i)x^{m(M)} \\ &= xy\sum_{M\in\mathcal{T}}\frac{1-y^{n(M)+1}}{1-y}x^{m(M)} \\ &= \frac{xy}{1-y}(t_0-yt) \end{aligned} \tag{10.24}$$

where $t=t(x,y)$ and $t_0=t(x,1)$.

Theorem 10.2 The enumerating function $t=t(x,y)$ of planar rooted maps satisfies the equation as

$$xy^2(1-y)t^2-(1-y+xy^2)t+xyt_0+(1-y)=0 \tag{10.25}$$

where $t_0=t(x,1)$.

Proof From (10.17),

$$t=f_{\mathcal{T}_0}(x,y)+f_{\mathcal{T}_1}(x,y)+f_{\mathcal{T}_2}(x,y).$$

Because $\mathcal{T}_0=\{\vartheta\}$ and ϑ has no edge, $f_{\mathcal{T}_0}(x,y)=1$. From (10.20) an (10.24),

$$t=1+xy^2t^2+\frac{xy}{1-y}(t_0-yt).$$

Via rearrangement of terms, (10.25) is soon found. □

Although (10.25) is a quadratic equation, because the occurrence of t_0 which is also unknown and the equation becomes an identity when $y=1$, complication occurs in solving the equation directly.

The discriminant of the equation (10.25), denoted by $D(x,y)$, is

$$\begin{aligned} D(x,y) &= (xy^2-y+1)^2-4(y-1)xy^2(y-1-xyt_0) \\ &= 1-2y+(1-2x)y^2+(x^2+2x \end{aligned}$$

$$-H(x))y^3 + H(x)y^4 \tag{10.26}$$

where

$$H(x) = 4x^2t_0 + x^2 - 4x. \tag{10.27}$$

Assume that $D(x, y)$ has the form as

$$\begin{aligned} D(x, y) &= (1 - \theta y)^2(1 + ay + by^2) \\ &= 1 - (2\theta - a)y + (\theta^2 - 2a\theta + b)y^2 \\ &\quad +\theta(a\theta - 2b)y^3 + \theta^2 by^4. \end{aligned} \tag{10.28}$$

By comparing with (10.26),

$$\theta = 1 + \frac{a}{2}, \tag{10.29}$$

and

$$\begin{cases} 1 - 2x = \theta(4 - 3\theta) + b; \\ x^2 + 2x - H(x) = \theta(2(\theta - 1)\theta - 2b); \\ H(x) = \theta^2 b. \end{cases} \tag{10.30}$$

Then, an equation about b with θ as the parameter is found as

$$\frac{1}{4}(1 - 4\theta + 3\theta^2 - b)^2 + 1 - 4\theta + 3\theta^2 - b - \theta^2 b$$
$$= 2(\theta - 1)\theta^2 - 2\theta b.$$

By rearrangement, it becomes

$$\begin{aligned} b^2 - (10\theta^2 - 16\theta + 6)b + (9\theta^4 - 32\theta^3 \\ +42\theta^2 - 24\theta + 5) = 0. \end{aligned} \tag{10.31}$$

The discriminant of (10.31) is

$$\begin{aligned} (10\theta^2 - 16\theta + 6)^2 - 4(9\theta^4 - 32\theta^3 + 42\theta^2 - 24\theta + 5) \\ = 64\theta^4 - 192\theta^3 + 208\theta^2 - 96\theta + 16 \\ = (8\theta^2 - 12\theta + 4)^2. \end{aligned}$$

Therefore,

$$b = (\theta - 1)^2, \text{or } 9\theta^2 - 14\theta + 5 = (9\theta - 5)(\theta - 1).$$

The latter has to be chosen in our case. By the last relation of (10.30),

$$H(x) = \theta^2(9\theta - 5)(\theta - 1).$$

From the first relation of (10.30) and (10.27), the expressions for x and b_0 with parameter θ are extracted as

$$\begin{cases} x = (3\theta - 2)(1 - \theta); \\ b_0 = \dfrac{4\theta - 3}{(3\theta - 2)^2}. \end{cases} \tag{10.32}$$

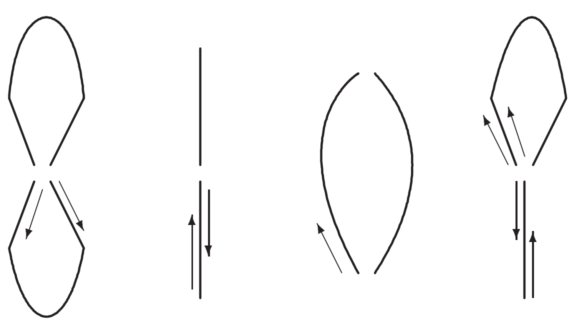
This enables us to get b_0 as a power series of x as

$$b_0(x) = \sum_{m \geq 0} \frac{2 \times 3^m (2m)!}{m!(m+2)!} x^m. \tag{10.33}$$

by eliminating the parameter θ via Lagrangian inversion. More about this method can be seen in the monograph[Liu8].

Example 10.2 For $m = 2$, it is known from (10.33) that the number of non-isomorphic planar rooted maps is the coefficient of x^2. That is 9.

Because there are 4 non-isomorphic planar maps of size 2 as shown in Fig.10.2. The arrows on the same map represent the roots of non-isomorphic rooted ones. Such as there are, respectively, 2, 2, 1 and 4 non-isomorphic rooted maps in (a), (b), (c) and (d) of Fig.10.2 to get 9 altogether.



(a) (b) (c) (d)

Fig.10.2 Planar rooted maps of two edges

X.3 Nonorientable equation

Let $\mathcal{N}_m$ be the set of nonorientable rooted maps of size m. Of course, $m \geq 1$. And, let $\mathcal{N}_m$ be partitioned into $\mathcal{N}_m^{(\mathrm{I})}$ and $\mathcal{N}_m^{(\mathrm{II})}$, *i.e.*,

$$\mathcal{N}_m = \mathcal{N}_m^{(\mathrm{I})} + \mathcal{N}_m^{(\mathrm{II})} \tag{10.34}$$

where $\mathcal{N}_m^{(\mathrm{I})} = \{N|N \in \mathcal{N}_m, N - a \text{ orientable}\}$ and

$$\mathcal{N}_m^{(\mathrm{II})} = \mathcal{N}_m - \mathcal{N}_m^{(\mathrm{I})} = \{N|N \in \mathcal{N}_m, N - a \text{ nonorientable}\},$$

$a = e_r(N)$ is still the rooted edge.

Lemma 10.8 Let $\mathcal{N}_m^{\langle\mathrm{I}\rangle} = \{N - a|\forall N \in \mathcal{N}_m^{(\mathrm{I})}\}$, then

$$\mathcal{N}_m^{\langle\mathrm{I}\rangle} = \mathcal{M}_{m-1} \tag{10.35}$$

where $\mathcal{M}_{m-1}$ is the set of orientable rooted maps of size $m-1$, $m \geq 1$.

Proof Because of the nonorientability of $N \in \mathcal{N}_m^{(\mathrm{I})}$ and the orientability of $N - a$, from Corollary 3.1, a is not a segmentation edge. Based on Theorem 3.4, $N - a$ is always an orientable map. So, for any $N \in \mathcal{N}_m^{\langle\mathrm{I}\rangle}$, $N \in \mathcal{M}_{m-1}$, $m \geq 1$. This implies $\mathcal{N}_m^{\langle\mathrm{I}\rangle} \subseteq \mathcal{M}_{m-1}$.

Conversely, for any $N = (\mathcal{X}, \mathcal{P}) \in \mathcal{M}_{m-1}$, by appending an edge a' on N, $N' = (\mathcal{X}', \mathcal{P}')$ is obtained where $\mathcal{X}' = \mathcal{X} + Kr'$ and $\mathcal{P}'$ is different from $\mathcal{P}$ only at the vertex

$$(r')_{\mathcal{P}'} = (r', \beta r', \langle r\rangle_{\mathcal{P}}).$$

Since Kr' is not a segmentation edge, from Theorem 3.7, N' is a map. And since $\beta r' \in \{r'\}_{\Psi'}, \Psi' = \Psi_{\{\mathcal{P}',\gamma\}}$, from Theorem 4.1, N' is nonorientable. Further, because $N = N' - a'$ is orientable and $m(N) + 1 = m(N') = m$, $N \in \mathcal{N}_m^{\langle\mathrm{I}\rangle}$. This implies $\mathcal{M}_{m-1} \subseteq \mathcal{N}_m^{\langle\mathrm{I}\rangle}$. $\square$

For any $M = (\mathcal{X}, \mathcal{P}) \in \mathcal{M}_m$, since M is orientable, assume

$$\{r\}_\Psi = \{r, \psi_1 r, \cdots, \psi_{2m-1} r\},$$

$\psi_i \in \Psi = \Psi_{\{\gamma,\mathcal{P}\}}$, $i = 1, 2, \cdots, 2m(M) - 1 = 2m - 1$. By appending the edge r',

$$A(M) = \{A_0(M), A_1(M), \cdots, A_{2m}(M)\}$$

is obtained where $A_i(M) = M + e_{r_i} = (\mathcal{X}_i, \mathcal{P}_i)$ such that $\mathcal{X}_i = \mathcal{X} + Kr_i$ and $\mathcal{P}_i$ is determined in the following manner:

$$\begin{cases} \beta r_0 \text{ in the angle } \langle \alpha\mathcal{P}^{-1}r, r\rangle \\ \beta r_i (i = 1, 2, \cdots, 2m(M) - 1) \text{ in the angle } \langle \alpha\mathcal{P}^{-1}\psi_i r, \psi_i r\rangle, \\ \beta r_{2m(M)} \text{ in the angle } \langle r, \alpha\mathcal{P}^{-1}r\rangle. \end{cases}$$

Because $\beta r_i \in \{r_i\}_{\Psi_i}$ where $\Psi_i = \Psi_{\{\gamma,\mathcal{P}_i\}}$, $i = 0, 1, \cdots, 2m(M)$, from Theorem 4.1, A_i are all nonorientable. Because $A_i(M) - e_{r_i} \in \mathcal{M}_m$, $A_i(M) \in \mathcal{N}^{(\mathrm{I})}_{m+1}$, $0 \leq i \leq 2m(M)$. From Lemma 10.8,

$$\mathcal{N}^{(\mathrm{I})}_{m+1} = \sum_{M \in \mathcal{M}_m} A(M) \tag{10.36}$$

$m \geq 0$. Of course, $\mathcal{M}_0$ consists of only the trivial map.

For $\mathcal{N}^{(\mathrm{II})}_m$, two cases should be considered: $\mathcal{N}^{(\mathrm{N})}_m$ and $\mathcal{N}^{(\mathrm{T})}_m$, *i.e.*,

$$\mathcal{N}^{(\mathrm{II})}_m = \mathcal{N}^{(\mathrm{N})}_m + \mathcal{N}^{(\mathrm{T})}_m \tag{10.37}$$

where

$$\mathcal{N}^{(\mathrm{N})}_m = \{N \mid \forall N \in \mathcal{N}^{(\mathrm{II})}_m, a = e_r \text{ is a terminal, or segmentation edge}\}$$

and

$$\mathcal{N}^{(\mathrm{T})}_m = \{N \mid \forall N \in \mathcal{N}^{(\mathrm{II})}_m, a \text{ is neither terminal nor segmentation edge}\}.$$

Of course, $\mathcal{N}^{(\mathrm{T})}_m = \mathcal{N}^{(\mathrm{II})}_m - \mathcal{N}^{(\mathrm{N})}_m$.

Lemma 10.9 Let $\mathcal{N}^{\langle\mathrm{N}\rangle}_m = \{N - a \mid \forall N \in \mathcal{N}^{(\mathrm{N})}_m\}$. Then,

$$\begin{aligned} \mathcal{N}^{\langle\mathrm{N}\rangle}_m = & \sum_{\substack{n_1+n_2=m-1 \\ n_1,n_2\geq 0}} \mathcal{M}_{n_1} \times \mathcal{N}_{n_2} + \sum_{\substack{n_1+n_2=m-1 \\ n_1,n_2\geq 0}} \mathcal{N}_{n_1} \times \mathcal{M}_{n_2} \\ & + \sum_{\substack{n_1+n_2=m-1 \\ n_1,n_2\geq 0}} \mathcal{N}_{n_1} \times \mathcal{N}_{n_2} \end{aligned} \tag{10.38}$$

where $\times$ represents the Cartesian product of sets.

Proof Easy to see except for noticing that $N - a$ has a transitive block which is the trivial map when a is a terminal edge. $\square$

Lemma 10.10 Let $\mathcal{N}_m^{\langle \mathrm{T} \rangle} = \{N - a | \forall N \in \mathcal{N}_m^{(\mathrm{T})}\}$. Then,

$$\mathcal{N}_m^{\langle \mathrm{T} \rangle} = \mathcal{N}_{m-1} \tag{10.39}$$

where $m \geq 2$.

Proof Because $a = e_r$ is neither terminal nor segmentation edge, from Theorem 3.4, $N \in \mathcal{N}_m^{\langle \mathrm{T} \rangle}$. By the nonorientability and the size $m - 1$, $N \in \mathcal{N}_{m-1}$, *i.e.*,

$$\mathcal{N}_m^{\langle \mathrm{T} \rangle} \subseteq \mathcal{N}_{m-1}.$$

On the other hand, for any $N = (\mathcal{X}, \mathcal{P}) \in \mathcal{N}_{m-1}$, we have $N' = (\mathcal{X}', \mathcal{P}')$ such that $\mathcal{X}' = \mathcal{X} + Kr'$ and $\mathcal{P}'$ is different from $\mathcal{P}$ only at the vertex $(r')_{\mathcal{P}'} = (r', \gamma r', \langle r \rangle_{\mathcal{P}})$. Since N is nonorientable and $a' = Kr'$ is neither terminal nor segmentation edge, $N' \in \mathcal{N}_m^{(\mathrm{T})}$. Thus, $N = N' - a' \in \mathcal{N}_m^{\langle \mathrm{T} \rangle}$. This implies

$$\mathcal{N}_{m-1} \subseteq \mathcal{N}_m^{\langle \mathrm{T} \rangle}.$$

In consequence, the lemma is proved. $\square$

One attention should be paid to is that when $m = 1$, there is only one nonorientable map $(Kr, (r, \beta r))$, and $(Kr, (r, \beta r)) \in \mathcal{N}^{(\mathrm{I})}$. Thus, (10.39) is meaningful only for $m \geq 2$.

On the basis of this lemma, it is necessary to see how many $N' \in \mathcal{N}_{m+1}^{(\mathrm{T})}$ can be produced from one $N \in \mathcal{N}_m$ such that $N = N' - a'$.

Because N is nonorientable, let $I = \{r, \psi_1 r, \psi_2 r, \cdots, \psi_{2m-1} r\}$ be consists of half the elements in $\{r\}_\Psi = \mathcal{X}$, $\Psi = \Psi_{\{\gamma, \mathcal{P}\}}$, such that for any $x \in I$, $Kx \cap I = \{x, \gamma x\}$. Two cases are now considered.

Case 1 For any $N = (\mathcal{X}, \mathcal{P}) \in \mathcal{N}_m$, let

$$B(N) = \{B_0(N), B_1(N), B_2(N), \cdots, B_{2m}(N)\}$$

where $B_j(N) = (\mathcal{X}_j, \mathcal{P}_j) = N + e_{r_j}$, $j = 0, 1, 2, \cdots, 2m$, have

$$\begin{cases} \beta r_0 \text{ in the angle } \langle \alpha \mathcal{P}^{-1} r, r \rangle, \\ \beta r_j (j = 1, 2, \cdots, 2m - 1) \text{ in the angle } \langle \alpha \mathcal{P}^{-1} \psi_j r, jr \rangle, \\ \beta r_{2m} \text{ in the angle } \langle r, \alpha \mathcal{P}^{-1} r \rangle. \end{cases}$$

Case 2 For any $N = (\mathcal{X}, \mathcal{P}) \in \mathcal{N}_m$, let

$$C(N) = \{C_0(N), C_1(N), C_2(N), \cdots, C_{2m}(N)\}$$

where $C_j(N) = (\mathcal{Y}_j, \mathcal{Q}_j) = N + e_{r_j}$, $j = 0, 1, 2, \cdots, 2m$, have

$$\begin{cases} \gamma r_0 \text{ in the angle } \langle \alpha\mathcal{P}^{-1}r, r\rangle, \\ \gamma r_j (j = 1, 2, \cdots, 2m-1) \text{ in the angle } \langle \alpha\mathcal{P}^{-1}\psi_j r, jr\rangle, \\ \gamma r_{2m} \text{ in the angle } \langle r, \alpha\mathcal{P}^{-1}r\rangle. \end{cases}$$

On the basis of Lemma 10.10, from the conjugate axiom,

$$\mathcal{N}_{m+1}^{(\mathrm{T})} = \sum_{N \in \mathcal{N}_m} (B(N) + C(N)) \tag{10.40}$$

for $m \geq 1$.

Because $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2 + \cdots$, the enumerating function

$$\begin{aligned} f_{\mathcal{N}}(x) &= \sum_{m \geq 1} (\sum_{N \in \mathcal{N}_m} 1) x^m = f_{\mathcal{N}^{(\mathrm{I})}}(x) + f_{\mathcal{N}^{(\mathrm{II})}}(x) \\ &= f_{\mathcal{N}^{(\mathrm{I})}}(x) + f_{\mathcal{N}^{(\mathrm{N})}}(x) + f_{\mathcal{N}^{(\mathrm{T})}}(x). \end{aligned} \tag{10.41}$$

Lemma 10.11 For $\mathcal{N}^{(\mathrm{I})} = \mathcal{N}_1^{(\mathrm{I})} + \mathcal{N}_2^{(\mathrm{I})} + \cdots$,

$$f_{\mathcal{N}^{(\mathrm{I})}}(x) = x f_{\mathcal{M}} + 2x^2 \frac{\mathrm{d} f_{\mathcal{M}}}{\mathrm{d}x} \tag{10.42}$$

where $f_{\mathcal{M}} = f_{\mathcal{M}}(x)$ is the enumerating function of orientable rooted maps determined by equation (10.14).

Proof On the basis of (10.35–36), from Lemma 9.10, the lemma is obtained . □

Lemma 10.12 For $\mathcal{N}^{(\mathrm{N})} = \mathcal{N}_1^{(\mathrm{N})} + \mathcal{N}_2^{(\mathrm{N})} + \cdots$,

$$f_{\mathcal{N}^{(\mathrm{N})}}(x) = 2x f_{\mathcal{M}} f_{\mathcal{N}} + x f_{\mathcal{N}}^2 \tag{10.43}$$

where $f_{\mathcal{M}} = f_{\mathcal{M}}(x)$ as in (10.10) and $f_{\mathcal{N}} = f_{\mathcal{N}}(x)$ as in (10.41).

Proof A direct result of Lemma 10.9. □

Lemma 10.13 For $\mathcal{N}^{(\mathrm{T})} = \mathcal{N}_1^{(\mathrm{T})} + \mathcal{N}_2^{(\mathrm{T})} + \cdots$,

$$f_{\mathcal{N}^{(\mathrm{T})}}(x) = 2xf_{\mathcal{N}} + 4x^2\frac{\mathrm{d}f_{\mathcal{N}}}{\mathrm{d}x}. \tag{10.44}$$

Proof On the basis of Lemma 10.10 with its extension (10.40), from Lemma 10.11, (10.44) is soon obtained. □

Theorem 10.3 The following equation about f

$$\begin{cases} 4x^2\dfrac{\mathrm{d}f}{\mathrm{d}x} = a(x)f - xf^2 - 2xb(x); \\ \dfrac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x=0} = 1 \end{cases} \tag{10.45}$$

where

$$\begin{cases} a(x) = 1 - 2x - 2xf_{\mathcal{M}}; \\ b(x) = f_{\mathcal{M}} - 2x\dfrac{\mathrm{d}f_{\mathcal{M}}}{\mathrm{d}x} \end{cases}$$

is well defined in the ring of power series with all coefficients non-negative integers and negative powers finite. And, the solution is $f = f_{\mathcal{N}}(x)$.

Proof Let $f = N_1x + N_2x^2 + N_3x^3 + \cdots$, then from (10.45) all the coefficients can be determined by the recursion

$$\begin{cases} N_m = (4m-2)N_{m-1} + (2m-1)F_{m-1} \\ \qquad +2\displaystyle\sum_{i=1}^{m-1} N_iF_{m-1-i} + \sum_{i=1}^{m-2} N_iN_{m-1-i}, \\ \qquad m \geq 2; \\ N_1 = 1, \end{cases} \tag{10.46}$$

where F_m, $m \geq 0$, are known in (10.15). Because all N_m, $m \geq 1$, determined by (10.46) are positive integers, the former statement is true. The latter is directly deduced from (10.41–44). □

X.4 Gross equation

Let $\mathcal{R}_m$ be the set of general(orientable and nonorientable) rooted maps with size m, $m \geq 0$. Of course, $\mathcal{R}_0$ consists of only the trivial map.

For $m \geq 1$, $\mathcal{R}_m$ is partitioned into two subsets $\mathcal{R}_m^{(\mathrm{N})}$ and $\mathcal{R}_m^{(\mathrm{T})}$, *i.e.*,

$$\mathcal{R}_m = \mathcal{R}_m^{(\mathrm{N})} + \mathcal{R}_m^{(\mathrm{T})} \tag{10.47}$$

where

$$\mathcal{R}_m^{(\mathrm{N})} = \{R|\forall R \in \mathcal{R}_m, e_r(R) \text{ is a terminal link} \\ \text{or segmentation edge}\}$$

and

$$\mathcal{R}_m^{(\mathrm{T})} = \{R|\forall R \in \mathcal{R}_m, e_r(R) \text{ is neither terminal} \\ \text{nor segmentation edge}\}.$$

Of course, $\mathcal{R}_m^{(\mathrm{T})} = \mathcal{R}_m - \mathcal{R}_m^{(\mathrm{N})}$.

Lemma 10.14 Let $\mathcal{R}_m^{\langle \mathrm{N} \rangle} = \{R - a|\forall R \in \mathcal{R}_m^{(\mathrm{N})}\}$, then

$$\mathcal{R}_m^{\langle \mathrm{N} \rangle} = \sum_{\substack{n_1+n_2=m-1 \\ n_1,n_2 \geq 0}} \mathcal{R}_{n_1} \times \mathcal{R}_{n_1}, \tag{10.48}$$

$m \geq 1$.

Proof For any $R \in \mathcal{R}_M^{(\mathrm{N})}$, because $a = e_r(R)$ is a terminal link or a segmentation edge, $R - a$ has two transitive block (when a is a terminal link, the trivial map is seen as a transitive block in its own right), $R - a = R_1 + R_2$ and $R_1 \in \mathcal{R}_{n_1}, R_2 \in \mathcal{R}_{n_2}$. In other words, the set on the left hand side of (10.48) is a subset of the set of its right.

Conversely, for any $R_1 = (\mathcal{X}_1, \mathcal{P}_1) \in \mathcal{R}_{n_1}$ and $R_2 = (\mathcal{X}_2, \mathcal{P}_2) \in \mathcal{R}_{n_2}$, by appending $a = e_r$, $R = (\mathcal{X}, \mathcal{P})$ is obtained where $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2 + Kr$ and $\mathcal{P}$ is different from $\mathcal{P}_1$ and $\mathcal{P}_2$ only at the vertices $(r)_{\mathcal{P}} = (r, \langle r_1 \rangle_{\mathcal{P}_1})$ and $(\gamma r)_{\mathcal{P}} = (\gamma r, \langle r_2 \rangle_{\mathcal{P}_2})$. It is easily checked that $R \in \mathcal{R}_m$, $m = n_1 + n_2 + 1$. In other words, the set on the right hand side of (10.48) is a subset of the set on the left. $\square$

Since $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1 + \mathcal{R}_2 + \cdots$, the enumerating function

$$\begin{aligned} f_{\mathcal{R}}(x) &= \sum_{m\geq 0}\Big(\sum_{R\in\mathcal{R}_m} 1\Big)x^m \\ &= f_{\mathcal{R}_0}(x) + f_{\mathcal{R}^{(\mathrm{N})}}(x) + f_{\mathcal{R}^{(\mathrm{T})}}(x), \end{aligned} \tag{10.49}$$

where $\mathcal{R}^{(\mathrm{N})} = \mathcal{R}_1^{(\mathrm{N})} + \mathcal{R}_2^{(\mathrm{N})} + \cdots$ and $\mathcal{R}^{(\mathrm{T})} = \mathcal{R}_1^{(\mathrm{T})} + \mathcal{R}_2^{(\mathrm{T})} + \cdots$.

First, because $\mathcal{R}_0$ consists of only the trivial map,

$$f_{\mathcal{R}_0}(x) = 1. \tag{10.50}$$

Then, from Lemma 10.14,

$$f_{\mathcal{R}^{(\mathrm{N})}}(x) = x f_{\mathcal{R}}^2, \tag{10.51}$$

where $f_{\mathcal{R}} = f_{\mathcal{R}}(x)$.

In order to evaluate $f_{\mathcal{R}^{(\mathrm{T})}}(x)$, $\mathcal{R}^{(\mathrm{T})}$ has to be decomposed.

Lemma 10.15 Let $\mathcal{R}_m^{\langle\mathrm{T}\rangle} = \{R - a \mid \forall R \in \mathcal{R}_m^{(\mathrm{T})}\}$, then

$$\mathcal{R}_m^{\langle\mathrm{T}\rangle} = \mathcal{R}_{m-1}, \tag{10.52}$$

where $m \geq 1$.

Proof For any $R' \in \mathcal{R}_m^{(\mathrm{T})}$, because $a' = e_{r'}$ is neither terminal link nor segmentation edge, from Theorem 3.4, $R = R' - a' \in \mathcal{R}_{m-1}$. This implies

$$\mathcal{R}_m^{\langle\mathrm{T}\rangle} \subseteq \mathcal{R}_{m-1}.$$

Conversely, for any $R = (\mathcal{X}, \mathcal{P}) \in R_{m-1}$, by appending the edge $a' = Kr'$, $R' = (\mathcal{X}', \mathcal{P}')$ is obtained where $\mathcal{X}' = \mathcal{X} + Kr'$ and $\mathcal{P}'$ is different from $\mathcal{P}$ only at the vertex $(r')_{\mathcal{P}'} = (r', \gamma r', \langle r\rangle_{\mathcal{P}})$. From Theorem 3.7, $R' \in \mathcal{R}_m$. Because a' is neither terminal link nor segmentation edge and $R = R' - a'$, $R \in \mathcal{R}_m^{\langle\mathrm{T}\rangle}$. This implies

$$\mathcal{R}_{m-1} \subseteq \mathcal{R}_m^{\langle\mathrm{T}\rangle}.$$

The lemma is proved. □

Based on this, what should be further considered for is how many $R' \in \mathcal{R}_{m+1}^{(\mathrm{T})}$ can be produced from one $R \in \mathcal{R}_m$ such that $R = R' - a'$.

Because R is a map(orientable, or nonorientable), let

$$I = \{r, \psi_1 r, \psi_2 r, \cdots, \psi_{2m-1} r\}$$

be the set of elements in correspondence with a primal trail code, or dual trail code. For any $x \in I$, $Kx \cap I = \{x, \gamma x\}$ has two possibilities as cases.

Case 1 For any $R = (\mathcal{X}, \mathcal{P}) \in \mathcal{R}_m$, let

$$D(R) = \{D_0(R), D_1(R), D_2(R), \cdots, D_{2m}(R)\}$$

where $D_j(R) = (\mathcal{X}_j, \mathcal{P}_j) = R + e_{r_j}$, $j = 0, 1, 2, \cdots, 2m$, have

$$\begin{cases} \beta r_0 \text{ in the angle } \langle \alpha\mathcal{P}^{-1} r, r\rangle, \\ \beta r_j (j = 1, 2, \cdots, 2m-1) \text{ in the angle } \langle \alpha\mathcal{P}^{-1}\psi_j r, jr\rangle, \\ \beta r_{2m} \text{ in the angle } \langle r, \alpha\mathcal{P}^{-1} r\rangle. \end{cases}$$

Cases 2 For any $R = (\mathcal{X}, \mathcal{P}) \in \mathcal{R}_m$, let

$$E(R) = \{E_0(R), E_1(R), E_2(R), \cdots, E_{2m}(R)\}$$

where $E_j(R) = (\mathcal{Y}_j, \mathcal{Q}_j) = R + e_{r_j}$, $j = 0, 1, 2, \cdots, 2m$, have

$$\begin{cases} \gamma r_0 \text{ in the angle } \langle \alpha\mathcal{P}^{-1} r, r\rangle, \\ \gamma r_j (j = 1, 2, \cdots, 2m-1) \text{ in the angle } \langle \alpha\mathcal{P}^{-1}\psi_j r, jr\rangle, \\ \gamma r_{2m} \text{ in the angle } \langle r, \alpha\mathcal{P}^{-1} r\rangle. \end{cases}$$

Based on Lemma 10.15, from the conjugate axiom,

$$\mathcal{R}^{(\mathrm{T})}_{m+1} = \sum_{R \in \mathcal{R}_m} (D(R) + E(R)) \tag{10.53}$$

for $m \geq 1$.

Because $\mathcal{R}^{(\mathrm{T})} = \mathcal{R}^{(\mathrm{T})}_1 + \mathcal{R}^{(\mathrm{T})}_2 + \cdots$, from Lemma 10.15 with its extension (10.53) and Lemma 9.10, the enumerating function

$$f_{\mathcal{R}^{(\mathrm{T})}}(x) = 2x f_{\mathcal{R}} + 4x^2 \frac{\mathrm{d} f_{\mathcal{R}}}{\mathrm{d}x} \tag{10.54}$$

Theorem 10.4 The equation about f

$$\begin{cases} 4x^2\dfrac{\mathrm{d}f}{\mathrm{d}x} = -1 + (1-2x)f - xf^2; \\ f_0 = f(0) = 1 \end{cases} \tag{10.55}$$

is well defined in the ring of power series with coefficients all nonnegative integers and terms of negative power finite. And, the solution is $f = f_{\mathcal{R}}(x)$.

Proof In virtue of the initial condition of equation (10.55), assume $f = R_0 + R_1x + R_2x^2 + \cdots$. Of course, $R_0 = f_0 = 1$. Further, from equation (10.55), the recursion

$$\begin{cases} R_m = (4m-2)R_{m-1} + \displaystyle\sum_{i=0}^{m-1} R_iR_{m-1-i}, \\ \qquad m \geq 1; \\ R_0 = 1 \end{cases} \tag{10.56}$$

is soon found for determine all the coefficients R_m, $m \geq 0$. It is easily checked that all of them are positive integers and hence the former statement is true.

The latter is a direct result of (10.50–51) and (10.53). □

X.5 The number of rooted maps

First, let $\sigma_m = (F_0, F_1, \cdots, F_m)$, $m \geq 0$, be the $m+1$ dimensional vector where F_m, $m \geq 0$, are the number of non-isomorphic orientable rooted maps with size m. And, $\sigma_M^{\mathrm{R}} = (F_m, F_{m-1}, \cdots, F_0)$ called the *reversed vector* of the vector σ_m. Easy to check that

$$\sigma_m^{\mathrm{TR}} = ((\sigma_m)^{\mathrm{T}})^{\mathrm{R}} = ((\sigma_m)^{\mathrm{R}})^{\mathrm{T}} = \sigma_m^{\mathrm{RT}} \tag{10.57}$$

where T the transposition of a matrix.

The recursion (10.15) for determining F_m, $m \geq 0$, becomes

$$\begin{cases} F_m = (2m-1)F_{m-1} + \sigma_{m-1}\sigma_{m-1}^{\mathrm{TR}}, \\ \qquad m \geq 1; \\ F_0 = 1. \end{cases} \tag{10.58}$$

By (10.58), the number of non-isomorphic orientable rooted maps with size m, $m \geq 1$, can be calculated. In the first column of Table 10.1, F_m, $m \leq 10$, are listed.

Then, let $\delta_m = (N_1, N_2, \cdots, N_1)$ where N_m is the number of non-isomorphic nonorientable rooted maps with size m for $m \geq 1$.

The recursion (10.46) for determining N_m, $m \geq 1$, becomes

$$\begin{cases} N_m = (4m-2)N_{m-1} + (2m-1)F_{m-1} \\ \qquad +2\delta_{m-1}\sigma_{m-2}^{\mathrm{TR}} + \delta_{m-2}\delta_{m-2}^{\mathrm{TR}}, \\ \qquad m \geq 2; \\ N_1 = 1 \end{cases} \tag{10.59}$$

where σ_{m-2} is given in (10.58).

By (10.59), the number N_m can be calculated for $m \geq 1$. In the second column of Table 10.1, N_m, $m \leq 10$, are listed.

Finally, let $\rho_m = (R_0, R_1, \cdots, R_m)$ where R_m is the number of non-isomorphic general maps with size m for $m \geq 0$.

The recursion (10.56) becomes

$$\begin{cases} R_m = (4m-2)R_{m-1} + \rho_{m-1}\rho_{m-1}^{\mathrm{TR}}, \\ \qquad m \geq 1; \\ R_0 = 1. \end{cases} \tag{10.60}$$

By (10.60), the number R_m can be calculated for $m \geq 0$. In the third column of Table 10.1, R_m, $m \leq 10$, are listed.

From those numbers in Table 10.1, it is also checked that the enumerating functions $f_{\mathcal{M}}(x)$, $f_{\mathcal{N}}(x)$ and $f_{\mathcal{R}}(x)$ of, respectively, non-isomorphic orientable, nonorientable and general(orientable and nonorientable) rooted maps with size as the parameter satisfy the relation as

$$f_{\mathcal{R}}(x) = f_{\mathcal{M}}(x) + f_{\mathcal{N}}(x). \tag{10.61}$$

m	F_m	N_m	R_m
0	1	0	1
1	2	1	3
2	10	14	24
3	74	223	297
4	706	4190	4896
5	8162	92116	100278
6	110410	2339894	2450304
7	1708394	67825003	69533397
8	29752066	2217740030	2247492096
9	576037442	80952028936	81528066378
10	12277827850	3268104785654	3280382613504

Table 10.1 Numbers of rooted maps with size less than 11

Activities on Chapter X

X.6 Observations

O10.1 Because a surface can be seen as a polygon with even edges pairwise identified in the plane, think that whether, or not, a map on a surface rather than plane can always represented by a planar one. If it can, explain the reason, or provide an example otherwise.

O10.2 For a rooted map M, let $m(M)$ and $l(M)$ be, respectively, the size and the valency of root-face in M, observe the numbers of non-isomorphic planar rooted maps for $m(M),\ l(M) \leq 3$.

Similarly, do the same with the valency of root-vertex $s(M)$ instead of $l(M)$.

A map with all its faces of 3-valent except for the root-face is called a *near triangulation*.

O10.3 For a near triangulation T, let $m(T)$ and $l(T)$ be, respectively, the size and the root-face valency of T. Observe the numbers of non-isomorphic planar rooted near triangulations for m, $l \leq 4$.

Similarly, do the same with the valency of root-vertex s instead of l.

A map with all its faces of valency 4 except for the root-face is called a *near quadrangulation*.

O10.4 For a near triangulation Q, let $m(Q)$ and $l(Q)$ be, respectively, the size and the root-face valency of TQ. Observe the numbers of non-isomorphic planar rooted near quadrangulation for m, $l \leq 5$.

Similarly, do the same with the valency of root-vertex s instead of l.

A planar map from which the result of deleting all the edges on a face is a tree is called a *Halin map*. The face is said to be *specific*.

O10.5 Evaluate the number of non-isomorphic rooted Halin maps with the root on the specific face by the parameters: the size m and the valency l of the specific face for $m \geq 6$ and $l \geq 3$.

O10.6 Try, by edge contraction, to determine the enumerating function of general rooted maps with the size as the parameter.

O10.7 Try, by edge contraction, to determine the enumerating function of rooted petal bundles with the size an the root-face valency as the two parameters.

O10.8 Try, by directly solving the equation (9.7), to determine the enumerating function of orientable rooted petal bundles h.

O10.9 Try, by directly solving the equation (9.30), to determine the enumerating function of nonorientable rooted petal bundles g.

O10.10 Observe the numbers of general Eulerian rooted maps with the size smaller.

X.7 Exercises

For a map, if the result of deleting all *inner vertices*(not *articulate vertex*, *i.e.*, a vertex of valency 1, or a *terminal*) of a spanning tree is, itself, a travel with only one vertex of valency probably greater than 2, then it is called a *pan-Halin map*. Because this travel is still a map and becomes a petal bundle via decreasing subdivision, such a petal bundle is called the *base map* of the pan-Halin map.

A pan-Halin map of which the base map is of size $2p$ and orientable genus $p \geq 0$, or of size q and nonorientable genus $q \geq 1$, is said to be *pre-standard* . If a pre-standard pan-Halin map has its base

map with each edge incident with at least one terminal of the tree, then it is said to be *standard*.

Let $\mathcal{H}_{\mathrm{psH}}$ be the set of all pre-standard pan-Halin rooted maps. The root r_H for $H \in \mathcal{H}_{\mathrm{psH}}$ is chosen be an element incident with the vertex and the face of the base map of H.

For any $H = (\mathcal{X}, \mathcal{P}) \in \mathcal{H}_{\mathrm{psH}}$, the tree T on H is seen as a *planted tree*(a plane tree with the root-vertex of valency 1) with its root $r_T = \mathcal{P}(\mathcal{P}\gamma)^t r_H$ where

$$t = \min\{s | (\mathcal{P}\gamma)^s r_H \text{ incident with a terminal of } T\}.$$

E10.1 Given the partition of vertices according to their valencies on a planted tree $\underline{j} = (j_1, j_2, \cdots,)$, *i.e.*, j_i, $i \geq 1$, is the number of unrooted vertices of valency i, prove that the number of non-isomorphic planted trees with the partition is

$$\frac{(n-1)!}{\underline{j}!}$$

where

$$n = 1 + \sum_{i \geq 1} j_i,$$

i.e., the order, and $\underline{j}! = \prod_{i \geq 1} j_i!$.

E10.2 Given the vertex partition $\underline{s} = (s_2, s_3, \cdots)$, prove that the number of non-isomorphic pre-standard pan-Halin rooted maps with the partition and their base maps of size m on a surface of orientable genus p is

$$2^m \binom{m+2p-1}{2p-1} \binom{s_3}{m} \frac{n!m}{\underline{s}!}$$

where

$$n + 2 = \sum_{i \geq 2} s_i, \text{ and } \underline{s}! = \prod_{s \geq 2} s_i!.$$

E10.3 Given the vertex partition $\underline{s} = (s_2, s_3, \cdots)$, Prove that the number of non-isomorphic pre-standard pan-Halin rooted maps

with the partition and their base maps of size m on a surface of nonorientable genus q is

$$2^m \binom{m+q-1}{q-1} \binom{s_3}{m} \frac{n!m}{\underline{s}!}$$

where

$$n+2 = \sum_{i\geq 2} s_i, \text{ and } \underline{s}! = \prod_{s\geq 2} s_i!.$$

E10.4 Given the vertex partition $\underline{s} = (s_2, s_3, \cdots)$, prove that the number of non-isomorphic standard pan-Halin rooted maps with the partition and their base maps of size m on a surface of orientable genus p is

$$2^m \binom{m-1}{2p-1} \binom{s_3}{m} \frac{n!m}{\underline{s}!}$$

where

$$n+2 = \sum_{i\geq 2} s_i,$$

and $\underline{s} \geq 0$, $\underline{s} \neq \underline{0}$, $m \geq 2p \geq 1$.

E10.5 Given the vertex partition $\underline{s} = (s_2, s_3, \cdots)$, prove that the number of non-isomorphic standard pan-Halin rooted maps with the partition and their base maps of size m on a surface of nonorientable genus q is

$$2^m \binom{m-1}{q-1} \binom{s_3}{m} \frac{n!m}{\underline{s}!}$$

where

$$n+2 = \sum_{i\geq 2} s_i,$$

and $\underline{s} \geq 0$, $\underline{s} \neq \underline{0}$, $m \geq q \geq 1$.

E10.6 Evaluate the number of near triangulations of size m on the projective plane.

E10.7 Evaluate the number of near triangulations of size m on the Klein bottle.

E10.8 Evaluate the number of rooted quadrangulations of size m on the projective plane.

E10.9 Evaluate the number of near quadrangulations of size m on the Klein bottle.

E10.10 Determine the enumerating function of rooted petal bundles with the size as the parameter on the torus.

E10.11 Determine the enumerating function of rooted petal bundles with the size as the parameter on the projective plane.

E10.12 Determine the enumerating function of orientable two vertex rooted maps with size as the parameter.

E10.13 Determine the enumerating function of nonorientable two vertex rooted maps with size as the parameter.

E10.14 Establish an equation satisfied by the enumerating function of general non-separable rooted maps.

E10.15 Establish an equation satisfied by the enumerating function of general Eulerian rooted maps.

A map is said to be *loopless* if its under graph has no self-loop.

E10.16 Establish an equation satisfied by the enumerating function of general loopless rooted maps.

A map is said to be *simple* if its under graph has neither self-loop nor multi-edge.

E10.17 Establish an equation satisfied by the enumerating function of general simple rooted maps.

X.8 Researches

R10.1 Given a relative genus $g \neq 0$(the case $g = 0$ is solved), determine the number of rooted near triangulations of size $m \geq |g|$ on

a surface of genus g.

A rooted map with all vertices of the same valency except for probably one vertex is said to be *near regular*. Among them, near 3-regular and near 4-regular are often encountered in literature.

Although near triangulations or near quadrangulations are, respectively, the dual maps of rear 3-regular, or near 4- regular maps, they are still considered for most convenience from a different point of view.

R10.2 Given a relative genus $g \neq 0$(the case $g = 0$ is solved), determine the number of rooted near 3-regular maps of size $m \geq |g|$ on a surface of genus g.

R10.3 Given a relative genus $g \neq 0$(the case $g = 0$ is solved), determine the number of rooted near quadrangulations of size $m \geq |g|$ on a surface of genus g.

R10.4 Given a relative genus $g \neq 0$(the case $g = 0$ is solved), determine the number of rooted near 4-regular maps of size $m \geq |g|$ on a surface of genus g.

R10.5 Given a relative genus $g \neq 0$(the case $g = 0$ is solved), determine the number of non-separable rooted maps of size $m \geq |g|$ on a surface of genus g.

R10.6 Given a relative genus $g \neq 0$(the case $g = 0$ is solved), determine the number of Eulerian rooted maps of size $m \geq |g|$ on a surface of genus g.

R10.7 Given a relative genus $g \neq 0$(the case $g = 0$ ia solved), determine the number of non-separable Eulerian rooted maps of size $m \geq |g|$ on a surface of genus g.

For the problems above, another parameter $l \geq 1$ is absolutely necessary in almost all cases. it is the valency of the extra vertex, or face according as the regularity is for vertices, or faces.

R10.8 Given a relative genus $g \neq 0$(the case $g = 0$ is known),

find a relation between general maps and quadrangulations on a surface of genus g.

R10.9 Given a relative genus $g \neq 0$(the case $g = 0$ is known), find a relation between general maps and triangulations on a surface of genus g.

R10.10 Given a relative genus $g \neq 0$(the case $g = 0$, a 1–to–1 correspondence between loopless planar rooted maps of size $m - 1$ and 2-connected planar rooted triangulations of $2m - 1$ unrooted faces should be found, but now unknown yet), find a relation between loopless rooted maps and triangulations on a surface of genus g.

R10.11 Present an expression of the solution for equation (10.14) by special functions, particularly the hyperbolic geometric function.

Chapter XI

Maps with Symmetry

- A relation between the number of rooted maps and the order of the automorphism group of a map is established.
- A general procedure is shown for determining the group order distribution of maps with given size via an example as an application of the relation.
- A principle for counting unrooted maps from rooted ones is provided.
- Based on the principle, a general procedure is shown for determining the genus distribution of unrooted maps with given size via two examples.
- Conversely, rooted maps can be also determined via unrooted maps.

XI.1 Symmetric relation

First, observe how to derive the number of non-isomorphic unrooted maps from that of non-isomorphic rooted maps when the automorphism group is known, or in other words, how to transform results without symmetry to those with symmetry.

Theorem 11.1 Let $n_0(\mathcal{U}; I)$ be the number of non-isomorphic rooted maps with a given set of invariants including the size in the set of maps $\mathcal{U}$ considered. If the order of automorphism group of each map M in $\mathcal{U}$ is independent of the map M itself, but only dependent on $\mathcal{U}$ and I, denoted by $\mathrm{aut}(\mathcal{U}; I)$, then the number of non-isomorphic unrooted maps with I in $\mathcal{U}$ is

$$n_1(\mathcal{U}; I) = \frac{\mathrm{aut}(\mathcal{U}; I) n_0(\mathcal{U}; I)}{4\epsilon} \tag{11.1}$$

where $\epsilon \in I$ is the size.

Proof Let map $M = (\mathcal{X}, \mathcal{P}) \in \mathcal{U}$. From Theorem 8.1, for any $x \in \mathcal{X}$,

$$|X_x| = |\{y| \; \exists \tau \in \mathrm{Aut}(M), y = \tau x\}| = \mathrm{aut}(M). \tag{11.2}$$

In view of Corollary 8.2, M itself produce

$$n_0(M) = \frac{|\mathcal{X}|}{|X_x|} = \frac{4\epsilon}{\mathrm{aut}(M)} \tag{11.3}$$

non-isomorphic rooted maps. Therefore, there are

$$\begin{aligned} n_0(\mathcal{U}; I) &= \sum_{M \in \mathcal{U}} \frac{4\epsilon}{\mathrm{aut}(M)} \\ &= \frac{4\epsilon}{\mathrm{aut}(\mathcal{U}; I)} n_1(\mathcal{U}; I) \end{aligned}$$

non-isomorphic rooted maps in $\mathcal{U}$. Via rearrangement, (11.1) is soon obtained. □

In chapter VIII, efficient algorithms are established for finding the automorphism group of a map, this enables us to get how many non-isomorphic rooted maps from a unrooted map by (11.1).

However, from Chapter IX and Chapter X, it is unnecessary to know the automorphism group for counting rooted maps. This enables us to enumerate unrooted maps via automorphism groups by employing (11.1).

Problem of type 1 For a set of maps $\mathcal{M}$ known the number of non-isomorphic rooted maps with a given size, determine the number of non-isomorphic unrooted maps with the given size according to the orders of their automorphism groups, or in other words, the distribution of unrooted maps on the orders of their automorphism groups.

Although this problem does not yet have general progress in present, a great amount of results for rooted case have already provided reachable conditions for the problem.

XI.2 An application

In what follows, provide a general procedure for solving the problem of type 1 via the determination of the distribution of rooted petal bundles on the orders of the automorphism groups of corresponding unrooted maps on the basis of Chapter IX.

From Table 9.1 at the end of Chapter IX, the number of non-isomorphic planar rooted petal bundles with size 4 is $H_4^{(0)} = 14$, shown in (a–n) of Fig.11.1.

In virtue of Corollary 8.2, the orders of their automorphism groups are possibly 1, 2, 4, 8 and 16 only 5 cases.

Case 1 $\mathrm{aut}(M) = 1$, $M = (Kx + Ky + Kz + Kt, \mathcal{J})$. None.

Case 2 $\mathrm{aut}(M) = 2$. 8 planar rooted petal bundles:

$$\mathcal{J}_1 = (x, y, \gamma y, \gamma x, z, \gamma z, t, \gamma t); \mathcal{J}_2 = (x, \gamma x, y, z, \gamma z, t, \gamma t, \gamma y);$$
$$\mathcal{J}_3 = (x, y, z, \gamma z, t, \gamma t, \gamma y, \gamma x);\ \mathcal{J}_4 = (x, y, \gamma y, z, \gamma z, \gamma x, t, \gamma t);$$
$$\mathcal{J}_5 = (x, y, z, \gamma z, \gamma y, t, \gamma t, \gamma x);\ \mathcal{J}_6 = (x, \gamma x, y, z, \gamma z, \gamma y, t, \gamma t);$$
$$\mathcal{J}_7 = (x, y, \gamma y, z, t, \gamma z, \gamma t, \gamma x);\ \mathcal{J}_8 = (x, \gamma x, y, \gamma y, z, t, \gamma t, \gamma z),$$

shown in (a–h) of Fig.11.1.

Case 3 $\quad \mathrm{aut}(M) = 4$. 4 planar rooted petal bundles:

$$\mathcal{J}_9 = (x, y, \gamma y, \gamma x, z, t, \gamma t, \gamma z); \mathcal{J}_{10} = (x, \gamma x, y, z, t, \gamma t, \gamma z, \gamma y);$$
$$\mathcal{J}_{11} = (x, y, z, t, \gamma t, \gamma z, \gamma y, \gamma x);\ \mathcal{J}_{12} = (x, y, z, \gamma z, \gamma y, \gamma x, t, \gamma t),$$

shown in (i–l) of Fig.11.1.

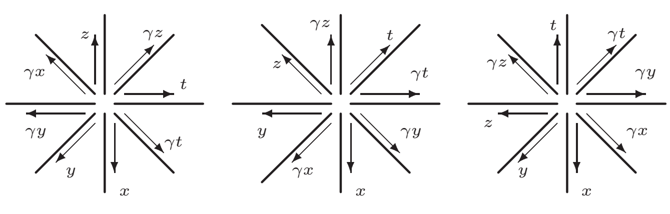
Case 4 $\quad \mathrm{aut}(M) = 8$. 2 planar rooted petal bundles:

$$\mathcal{J}_{13} = (x, \gamma x, y, \gamma y, z, \gamma z, t, \gamma t);\ \mathcal{J}_{14} = (x, y, \gamma y, z, \gamma z, t, \gamma t, \gamma x).$$

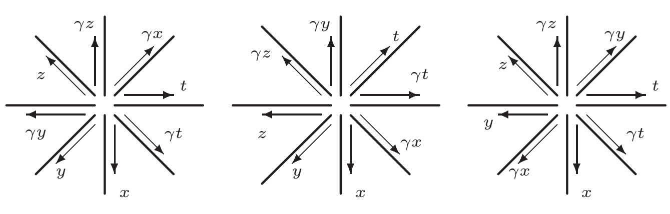
shown in (m, n) of Fig.11.1.

Case 5 $\quad \mathrm{aut}(M) = 16$. None.

This procedure can be done for determining the automorphism groups of a unrooted maps via their primal trail codes, or dual trail codes, by computers and then via the collection of the same class of them according to the orders of the groups.



(a) (b) (c)



(d) (e) (f)

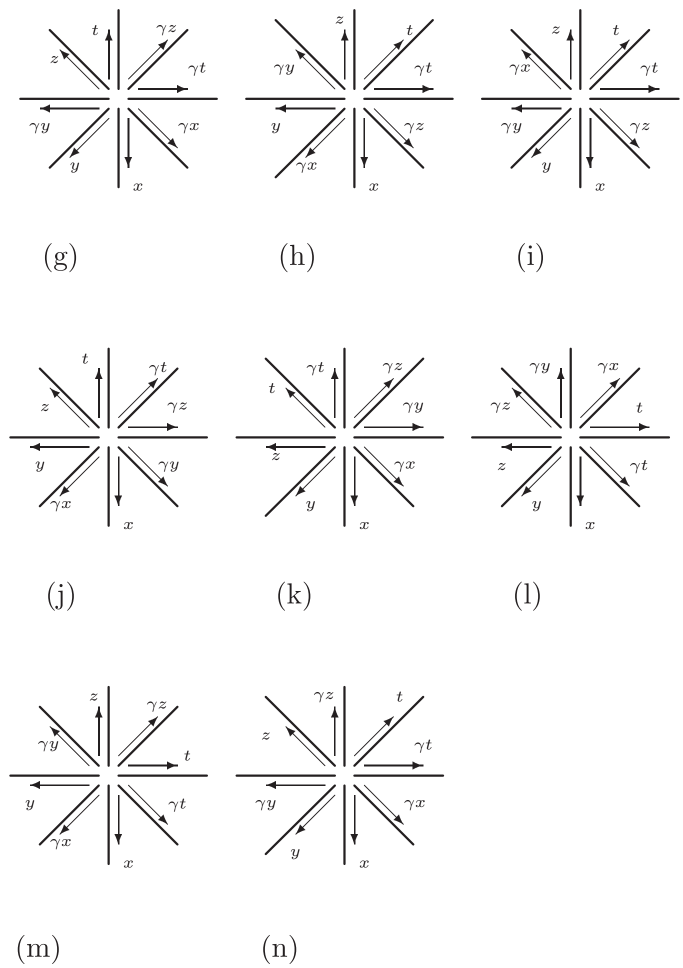


Fig.11.1 Planar petal bundles of size 4

XI.3 Symmetric Principle

Whenever the distribution of rooted maps on the orders of automorphism groups is given for a set of maps, the number of non-isomorphic unrooted maps can be soon extracted

Theorem 11.2 Let $n_{0i}(\mathcal{M}; I)$ be the number of non-isomorphic rooted maps with the set of invariants I and the order of their automorphism groups i in a set of maps $\mathcal{M}$ for $i|4\epsilon$, $1 \leq i \leq 4\epsilon$, where ϵ is

the size, then the number of non-isomorphic unrooted maps in $\mathcal{M}$ is

$$n_1(\mathcal{M}; I) = \sum_{\substack{i|4\epsilon \\ 1\leq i\leq 4\epsilon}} \frac{in_{0i}(\mathcal{M}; I)}{4\epsilon}. \tag{11.4}$$

Proof Let $n_{1i}(\mathcal{M}; I)$ be the number of non-isomorphic unrooted maps with the set of invariants I and the order of their automorphism groups i in the set of maps $\mathcal{M}$ for $i|4\epsilon$, $1 \leq i \leq 4\epsilon$, where ϵ is the size, then

$$n_1(\mathcal{M}; I) = \sum_{\substack{i|4\epsilon \\ 1\leq i\leq 4\epsilon}} n_{1i}(\mathcal{M}; I). \tag{11.5}$$

From Theorem 11.1, each unrooted map $M \in \mathcal{M}$, $\mathrm{Aut}(M) = i$, produces

$$\frac{4\epsilon}{i}$$

non-isomorphic rooted maps. Therefore,

$$n_{1i}(\mathcal{M}; I) = \frac{in_{0i}(\mathcal{M}; I)}{4\epsilon}. \tag{11.6}$$

By substituting (11.6) into (11.5), (11.4) is soon obtained. □

On the choice of the set of invariants I, two types should be mentioned. One is that the set I consists of only the size and the genus for determining the genus distribution of non-isomorphic maps in a set of maps $\mathcal{M}$. The other is that the set I consists of only the size and the orders of automorphisms for determining the symmetric distribution of non-isomorphic maps in a set of maps $\mathcal{M}$.

Problem of type 2 For a set of maps $\mathcal{M}$ with the number of non-isomorphic maps given, determine the number of non-isomorphic under graphs of maps in $\mathcal{M}$.

Although the justification of whether, or not, two graphs are isomorphic is much far from easy, a feasible approach to it is presented from the above discussion. Because the under graphs are isomorphic

if the two maps are isomorphic, the only thing we have to do is to classify non-isomorphic maps by their isomorphic under graphs.

On the other hand, for a graph, it is also possible to discuss how many non-isomorphic rooted maps are with the graph as their under graph, and then to discuss how many non-isomorphic unrooted maps are with the graph as their under graphs, and finally to classify maps according to the isomorphism of their under graphs.

XI.4 General examples

On the basis of the 15 orientable rooted petal bundles of size 3 and the 9 nonorientable rooted petal bundles of size 2(in Table 9.1 at the end of Chapter IX), a general procedure is established for determining the genus distribution of them.

Orientable case Let $M = (Kx + Ky + Kz, \mathcal{J}_i)$, $1 \leq i \leq 15$.

genus 0 5 orientable rooted petal bundles shown in (a–e) of Fig.11.2. Here,

$$\mathcal{J}_1 = (x, \gamma x, y, \gamma y, z, \gamma z);\ \mathcal{J}_2 = (x, y, \gamma y, z, \gamma z, \gamma x)$$

with the order of its automorphism group $\text{aut}(M) = 6$ are one unrooted map;

$$\mathcal{J}_3 = (x, \gamma x, y, z, \gamma z, \gamma y);\ \mathcal{J}_4 = (x, y, \gamma y, \gamma x, z, \gamma z);$$
$$\mathcal{J}_5 = (x, y, z, \gamma z, \gamma y, \gamma x)$$

with the order of its automorphism group $\text{aut}(M) = 4$ are one unrooted map.

Genus 1 10 orientable petal bundles shown in (f–o) of Fig.11.2. Here,

$$\mathcal{J}_6 = (x, y, \gamma x, z, \gamma z, \gamma y);\ \mathcal{J}_7 = (x, y, z, \gamma z, \gamma x, \gamma y);$$
$$\mathcal{J}_8 = (x, y, \gamma y, z, \gamma x, \gamma z);\ \mathcal{J}_9 = (x, \gamma x, y, z, \gamma y, \gamma z);$$
$$\mathcal{J}_{10} = (x, y, z, \gamma y, \gamma z, \gamma x);\ \mathcal{J}_{11} = (x, y, \gamma x, \gamma y, z, \gamma z)$$

with the order of its automorphism group $\mathrm{aut}(M) = 2$ are one unrooted map;

$$\mathcal{J}_{12} = (x, , y, z, \gamma y, \gamma x, \gamma z);\ \mathcal{J}_{13} = (x, y, \gamma x, z, \gamma y, \gamma z);$$
$$\mathcal{J}_{14} = (x, y, z, \gamma x, \gamma z, \gamma y)$$

with the order of its automorphism group $\mathrm{aut}(M) = 4$ are one unrooted map; and

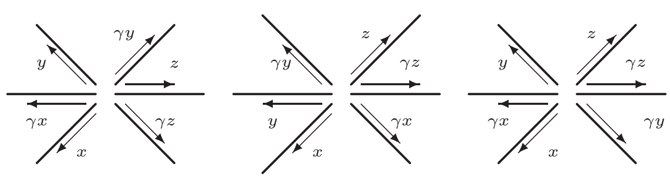
$$\mathcal{J}_{15} = (x, y, z, \gamma x, \gamma y, \gamma z)$$

with the order of its automorphism group $\mathrm{aut}(M) = 12$ is one unrooted map itself.

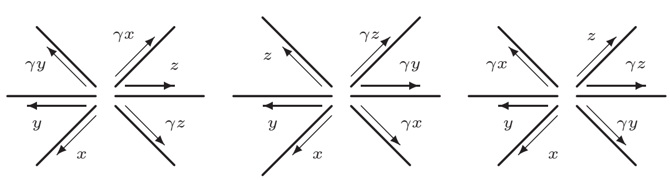
All are listed in Table 11.1 shown the genus distribution, group order distribution as well, of orientable unrooted maps.

Genus	aut(M) 1	2	3	4	6	12	Dist.
0	0	0	0	1	1	0	2
1	0	1	0	1	0	1	3
Dist.	0	1	0	2	1	1	5

Table 11.1 Distributions of orientable petal bundles



(a) (b) (c)



(d) (e) (f)

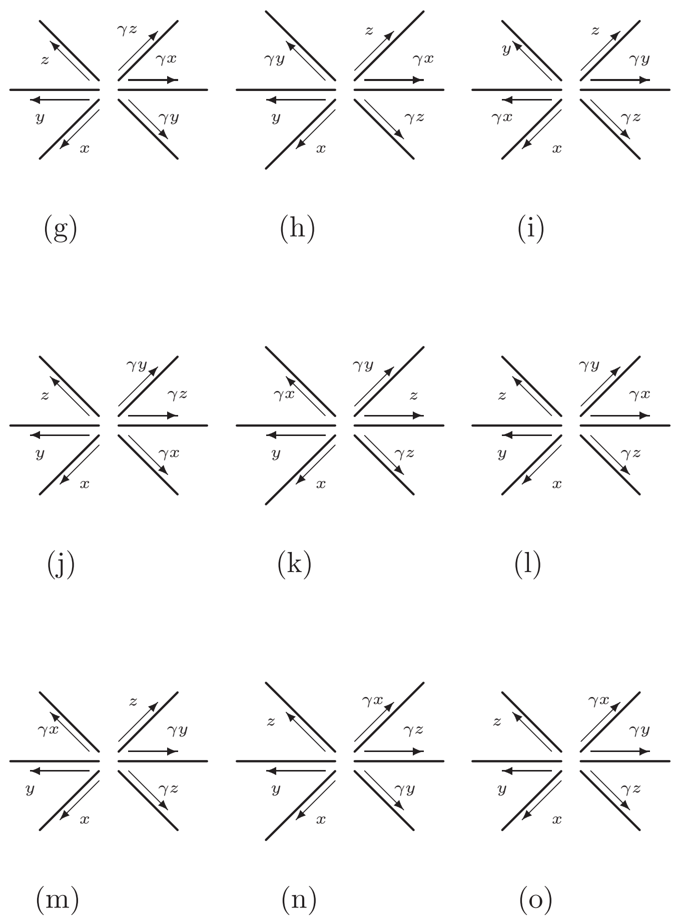


Fig.11.2 Orientable petal bundles of size 3

Nonorientable case Let $N = (Kx + Ky, \mathcal{J}_i)$, $1 \leq i \leq 9$.

Genus $\tilde{1}$ 5 nonorientable rooted petal bundles shown in Fig.9.4(e, a,c) and in Fig.9.3(a,c). Here,

$$\mathcal{J}_1 = (x, \beta y, \beta x, y)$$

with the order of its automorphism group $\text{aut}(N) = 8$ is one unrooted map itself;

$$\mathcal{J}_2 = (x, \beta x, y, \gamma y);\ \mathcal{J}_3 = (x, \beta x, \gamma y, y);$$
$$\mathcal{J}_4 = (x, \gamma x, y, \beta y);\ \mathcal{J}_5 = (x, \gamma x, \beta y, y)$$

with the order of its automorphism group $\mathrm{aut}(N) = 2$ are one unrooted map.

Genus $\tilde{2}$ 4 nonorientable rooted petal bundles shown in Fig.9.3(b) and Fig.9.4(b,d,f). Here,

$$\mathcal{J}_6 = (x, \beta y, \gamma x, y);\ \mathcal{J}_7 = (x, \gamma y, \beta x, y);$$
$$\mathcal{J}_8 = (x, \beta x, y, \beta y);\ \mathcal{J}_9 = (x, \beta x, \beta y, y)$$

with the order of its automorphism group $\mathrm{aut}(N) = 4$ are 2 unrooted maps.

All are listed in Table 11.2 shown the genus distribution, group order distribution as well, of nonorientable unrooted maps.

Genus	aut(M) 1	2	4	8	Dist.
$\tilde{1}$	0	1	0	1	2
$\tilde{2}$	0	0	2	0	2
Dist.	0	1	2	1	4

Table 11.2 Distributions of nonorientable petal bundles

Activities on Chapter XI

XI.5 Observations

O11.1 Given all the 54 planar rooted maps of size 3, find their distribution according to the orders of automorphism groups.

O11.2 Given all the 40 outerplanar rooted maps(the root incident with the outer face) of size 3, find their distribution according to the orders of automorphism groups.

O11.3 Observe how many non-isomorphic rooted maps whose under graph is K_4, *i.e.*, the complete graph of order 4.

O11.4 Observe the distribution of all the rooted maps whose under graph is K_4 according to the orders of their automorphism groups.

O11.5 Observe the genus distribution of all orientable maps whose under graph is K_4.

O11.6 Observe the genus distribution of all nonorientable maps whose under graph is K_4.

O11.7 Given all the 56 planar Euler rooted maps of size 4, find how many non-isomorphic unrooted maps among them.

O11.8 Given all the 27 planar 4-regular rooted maps of co-order 4, find how many non-isomorphic unrooted maps among them.

O11.9 For $m \geq 5$, try to determine the number of planar unrooted petal bundles of size m.

O11.10 For $m \geq 4$, try to determine the number of orientable

unrooted petal bundles of size m.

O11.11 For $m \geq 3$, try to determine the number of nonorientable unrooted petal bundles of size m.

XI.6 Exercises

E11.1 Prove that the number of non-isomorphic outerplanar rooted maps(the root is on the outer face) of size m is

$$\frac{2^m(2m)!}{(m+1)!m!}$$

for $m \geq 1$.

E11.2 Prove that the number of planar 4-regular rooted maps of co-order $n+1$ is

$$3^{n-1}\frac{2(2n-2)!}{(n+1)!(n-1)!}$$

for $n \geq 1$.

E11.3 Prove that the number of non-isomorphic planar rooted maps of size m is

$$\frac{2 \times 3^m}{(n+1)(n+2)}\binom{2m}{m}$$

for $m \geq 0$.

E11.4 Prove that the number of planar loopless rooted triangulations of size $3m$ is

$$\frac{2^{m+1}(3m)!}{m!(2m+2)!}$$

for $m \geq 1$.

E11.5 Prove that the number of non-isomorphic planar Euler rooted maps of size m is

$$\frac{3 \times 2^{m-1}(2m)!}{m!(m+2)!}$$

for $m \geq 1$.

E11.6 Prove that the number of non-isomorphic planar non-separable rooted maps of order p and co-order q is

$$\frac{(2p+q-5)!(2q+p-5)!}{(p-1)!(q-1)!(2p-3)!(2q-3)!}$$

where $p,\ q \geq 2$.

E11.7 Prove that the number of non-isomorphic planar simple rooted maps of size m is

$$\sum_{i=0}^{m-2} \frac{4(2m+1)!(2m-i-4)!}{i!(m-i-2)!(2m-i+1)!m!}$$

where $m \geq 2$.

E11.8 Prove that the number of non-isomorphic planar 3-connected rooted maps of size $m \geq 6$ is

$$(-1)^m 2 + R_{m-1}$$

where R_m, $m \geq 2$, are determined by the recursion

$$R_m = \frac{(7m-22)R_{m-1} + 2(2m-1)R_{m-2}}{2m},\ m \geq 3$$

with the initial conditions $R_1 = -1$ and $R_2 = 2$.

E11.9 For $m \geq 4$, determine the genus distribution of orientable rooted petal bundles with size m.

E11.10 For $m \geq 5$, determine the genus distribution of nonorientable rooted petal bundles with size m.

E11.11 For $m \geq 4$, determine the number of non-isomorphic outerplanar unrooted maps with size m.

XI.7 Researches

R11.1 Determine the number of non-isomorphic planar 4-regular unrooted maps of co-order $n+1$ for $n \geq 1$.

R11.2 Determine the number of non-isomorphic planar loopless unrooted triangulations of size $3m$, $m \geq 2$.

R11.3 Determine the number of non-isomorphic planar Euler unrooted maps of size $m \geq 2$.

R11.4 Determine the number of non-isomorphic planar nonseparable unrooted maps of size $m \geq 2$.

R11.5 Prove, or disprove, the conjecture that almost all trees have the order of their automorphism group 1 when the size is greater enough.

R11.6 Prove, or disprove, the conjecture that almost all maps with a given relative genus have the order of their automorphism group 1 when the size is large enough.

R11.7 Prove , or disprove, the conjecture that for a positive integer $g|\epsilon$, $g \geq 2$, almost no orientable map is with the order of automorphism group g when ϵ is large enough.

R11.8 Prove , or disprove, the conjecture that for a positive integer $g|\epsilon$, $g \geq 2$, almost no nonorientable map is with the order of automorphism group g when ϵ is large enough.

R11.9 Determine the genus distribution of 4-regular rooted maps of co-order $n + 1$, $n \geq 1$.

R11.10 Determine the genus distribution of loopless rooted triangulations of size $3m$, $m \geq 2$.

R11.11 Determine the genus distribution of Euler rooted map with size $m \geq 2$.

R11.12 Determine the genus distribution of nonseparable rooted map with size $m \geq 2$.

Although corresponding problems about genus distribution can also posed for unrooted case, they would be only suitable after the solution of rooted case in general.

Moreover, the genus distributions of maps with under graphs in some chosen classes can also be investigated.

Chapter XII

Genus Polynomials

- The set of associate surfaces of a graph are constructed to determine all of its distinct embeddings, or its super maps as well.
- A layer division of an associate surface of a graph is defined for establishing an operation to transform this surface into another associate surface. A procedure can be constructed for listing all other associate surfaces from an associate surface by this operation without repetition.
- A principle of determining the genus polynomial, called *handle polynomial*, of a graph is provided for the orientable case.
- The genus polynomial of a graph for nonorientable case, also called *crosscap polynomial*, is derived from the handle polynomial of the graph.

XII.1 Associate surfaces

Given a graph $G = (V, E)$ and a spanning tree T, the edge set E is partitioned into E_T(tree edge) and $\bar{E}_T$(cotree edge), *i.e.*, $E = E_T + \bar{E}_T$. Let $\bar{E}_T = \{i|i = 1, 2, \cdots, \beta\}$, $\beta = \beta(G)$ be the Betti number(or cyclic number) of G. If $i = (u[i], v[i])$, then i_u and i_v are, respectively, meant the semi-edges of i incident with $u[i]$ and $v[i]$.

Write $G' = (V + V_1, E_T + E_1)$, where $V_1 = \{v_i, \bar{v}_i | 1 \leq i \leq \beta\}$ and $E_1 = \{(u[i], v_i), (v[i], \bar{v}_i) | 1 \leq i \leq \beta\}$. Because G' is a tree itself, G' is called an *expanded tree* of T on G, and denoted by $\hat{T}_G$, or $\hat{T}$ in general case[Liu13–14].

Let $\delta = (\delta_1, \delta_2, \cdots, \delta_\beta)$ be a binary vector, or as a binary number of β digits. Denoted by $\hat{T}^\delta$ that $\hat{T}$, edges $(u[i], v_i)$ and $(v[i], \bar{v}_i)$ are labelled by i with indices: $+$(always omitted) or $-$, $1 \leq i \leq \beta$, where $\delta_i = 0$ means that the two indices are the same; otherwise, different. Then, δ is called an *assignment* of indices on $\hat{T}$.

For $v \in V$, let σ_v be a rotation at v and $\sigma_G = \{\sigma_v | \forall v \in V\}$, the rotation of G, then $\hat{T}_\sigma$ determine an embedding of $\hat{T}$ on the plane.

Theorem 12.1 For any σ as a rotation and δ as an assignment of indices, $\hat{T}_\sigma^\delta$ determines a joint tree.

Proof By the definition of a joint tree, it is soon seen. □

According to the theory described in Chapter 1, the orientability and genus are naturally defined to be that of its corresponding embedding.

Lemma 12.1 Joint tree $\hat{T}_\sigma^\delta$ is orientable if, and only if, $\delta = 0$.

Proof Because $\delta = 0$ implies each label with its two occurrences of different indices, the lemma is true. □

On a joint tree $\hat{T}_\sigma^\delta$, the surface determined by the boundary of the infinite face on the planar embedding of $\hat{T}_\sigma$ with δ on label indices is said to be an *associate*.

Lemma 12.2 The genus of a joint tree $\hat{T}_\sigma^\delta$ is that of its associate surface.

Proof Only from the definition of orientability of a joint tree.□

Two associate surfaces are the *same* is meant that they have the same assignment with the same cyclic order. Otherwise, *distinct*. Let

$\mathcal{F}(\beta)$ be the set of distinct surfaces on $I_\beta = \{1, 2, \cdots, \beta\}$.

For a surface $F \in \mathcal{F}(\beta)$ and a tree T on a graph G, if there exists an joint tree $\hat{T}^\delta_\sigma$ such that F is its associate surface, then F is said to be *admissible* . Let $\mathcal{F}_T(\beta)$ be the set of all distinct associate surfaces.

Given two integers p, $p \geq 0$, and q, $q \geq 1$, let $\mathcal{F}_T(\beta; p)$(or $\mathcal{F}_T(\beta; q)$, $q \geq 1$), $p \geq 0$, be all distinct admissible surfaces of orientable genus p(or nonorientable genus q).

Theorem 12.2 For any integer $p \geq 0$(or $q \geq 1$), the cardinality $|\mathcal{F}_T(\beta; p)|$(or $|\mathcal{F}_T(\beta; q)|$) is independent of the choice of tree T on G. Further, it is the number of distinct embeddings of G on a surfaces of orientable genus p(or nonorientable genus q).

Proof According to O1.14, a 1-to-1 correspondence between two sets of embeddings generated by two distinct spanning trees can be found such that same embeddings are in correspondence. This implies the theorem. □

Because of

$$|\mathcal{F}_T(\beta)| = \sum_{p\geq 0} |\mathcal{F}_T(\beta; p)| + \sum_{q\geq 1} |\mathcal{F}_T(\beta; q)|,$$

the following conclusion is found from the theorem.

Corollary 12.1 The cardinality $|\mathcal{F}_T(\beta)|$ is independent of the choice of tree T on G. Further, it is the number of distinct embeddings of G.

From Lemma 12.1, the nonorientability of an associate surface can be easily justified by only checking if it has a label i with the same index, *i.e.*, $\delta(i) = 1$.

Theorem 12.3 There is a 1-to-1 correspondence between associate surfaces and embeddings of a graph.

Proof First, we can easily seen that each embedding determines an associate surface. Then, we show that each associate surface is

determined by an embedding. Because of Theorem 12, this statement is derived. □

From what is mentioned above, it is soon seen that the problem of determining the genus distribution of all embeddings for a graph is transformed into that of finding the number of all distinct admissible associate surfaces in each elementary equivalent class and the problem on minimum and maximum genus of a graph is that among all admissible associate surfaces of the graph. All of them are done on a polygon.

XII.2 Layer division of a surface

Given a surface $S = (A)$. it is divided into segments layer by layer as in the following.

The 0th layer contains only one segment, *i.e.*, $A(= A_0)$.

The 1st layer is obtained by dividing the segment A_0 into l_1 segments, *i.e.*, $S = (A_1, A_2, \cdots, A_{l_1})$, where $A_1, A_2, \cdots, A_{l_1}$ are called the 1*st layer segments*.

Suppose that on $k-1$st layer, the $k-1$st layer segments are $A_{\underline{n}_{(k-1)}}$ where $\underline{n}_{(k-1)}$ is an integral $k-1$-vector satisfied by

$$\underline{1}_{(k-1)} \leq (n_1, n_2, \cdots, n_{k-1}) \leq \underline{N}_{(k-1)}$$

with $\underline{1}_{(k-1)} = (1, 1, \cdots, 1)$,

$$\underline{N}_{(k-1)} = (N_1, N_2, \cdots, N_{k-1}),$$

$N_1 = l_1 = N_{(1)}$, $N_2 = l_{A_{\underline{N}_{(1)}}}$, $N_3 = l_{A_{\underline{N}_{(2)}}}$, $\cdots$, $N_{k-1} = l_{A_{\underline{N}_{(k-2)}}}$, then the kth layer segments are obtained by dividing each $k-1$st layer segment as

$$A_{\underline{n}_{(k-1)},1}, A_{\underline{n}_{(k-1)},2}, \cdots, A_{\underline{n}_{(k-1)},l_{A_{\underline{n}_{(k-1)}}}} \tag{12.1}$$

where

$$\underline{1}_{(k)} = (\underline{1}_{(k-1)}, 1) \leq (\underline{n}_{(k-1)}, i) \leq \underline{N}_{(k)} = (\underline{N}_{(k-1)}, N_k)$$

and $N_k = l_{A_{\underline{N}_{(k-1)}}}$, $1 \le i \le N_k$. Segments in (I.1) are called *sons* of $A_{\underline{n}_{(k-1)}}$. Conversely, $A_{\underline{n}_{(k-1)}}$ is the *father* of any one in (12.1).

A layer segment which has only one element is called an *end segment* and others, *principle segments*.

For an example, let

$$S = (1, -7, 2, -5, 3, -1, 4, -6, 5, -2, 6, 7, -3, -4).$$

Fig.12.1 shows a layer division of S and Tab.12.1, the principle segments in each layer.

Layers	Principle segments
0th layer	$A = \langle 1, -7, 2-5; 3, -1, 4, -6, 5; -2, 6, 7, -3-4 \rangle$
1st layer	$B = \langle 1; -7, 2; -5 \rangle, C = \langle 3, -1; 4, -6; 5 \rangle,$ $D = \langle -2, 6; 7; -3, -4 \rangle$
2nd layer	$E = \langle -7; 2 \rangle, F = \langle 3; -1 \rangle, G = \langle 4; -6 \rangle,$ $H = \langle -2; 6 \rangle, I = \langle -3; -4 \rangle$

Tab.12.1 Layers and principle segments

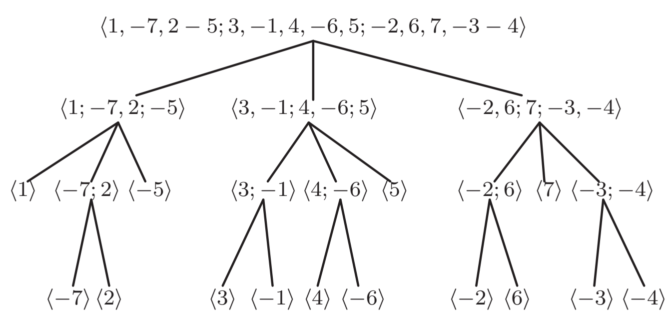


Fig.12.1 A layer division of S

For a layer division of a surface, if principle segments are dealt with vertices and edges are with the relationship between father and son, then what is obtained is a tree denoted by T. On T, by adding cotree edges as end segments, a graph $G = (V, E)$ is induced. For example, the graph induced from the layer division shown in Fig.12.1 is as

$$V = \{A, B, C, D, E, F, G, H, I\} \tag{12.2}$$

and
$$E = \{a, b, c, d, e, f, g, h, 1, 2, 3, 4, 5, 6, 7\}, \tag{12.3}$$
where
$$a = (A, B), b = (A, C), c = (A, D), d = (B, E),$$
$$e = (C, F), f = (C, G), g = (D, H), h = (D, I),$$
and
$$1 = (B, F), 2 = (E, H), 3 = (F, I), 4 = (G, I),$$
$$5 = (B, C), 6 = (G, H), 7 = (D, E).$$
By considering $E_T = \{a, b, c, d, e, f, g, h\}$, $\bar{E}_T = \{1, 2, 3, 4, 5, 6, 7\}$, $\delta_i = 0, i = 1, 2, \cdots, 7$, and the rotation σ implied in the layer division, a joint tree $\hat{T}^\delta_\sigma$ is produced.

Theorem 12.4 A layer division of a surface determines a joint tree. Conversely, a joint tree determines a layer division of its associate surface.

Proof From the procedure of constructing a layer division, a joint tree is determined. Conversely, it is natural. □

Then, an operation on a layer division is discussed for transforming an associate surface into another in order to visit all associate surfaces without repetition.

A layer segment with all its successors is called a *branch* in the layer division. The operation of interchanging the positions of two layer segments with the same father in a layer division is called an *exchanger*.

Lemma 12.3 A layer division of an associate surface of a graph under an exchanger is still a layer division of another associate surface. Conversely, the later under the same exchanger becomes the former.

Proof From the correspondence between layer divisions and associate surfaces, the lemma can be obtained. □

On the basis of this lemma, exchanger can be seen as an operation on the set of all associate surfaces of a graph.

Lemma 12.4 The exchanger is closed in the set of all associate surfaces of a graph.

Proof From the correspondence between joint trees and layer divisions, the conclusion of the lemma is seen. □

Lemma 12.5 Let $\mathcal{A}(G)$ be the set of all associate surfaces of a graph G, then for any S_1, $S_2 \in \mathcal{A}(G)$, there exist a sequence of exchangers on the set such that S_1 can be transformed into S_2.

Proof By considering the joint trees and layer divisions, the lemma is right. □

If $\mathcal{A}(G)$ is dealt as the vertex set and an edge as an exchanger, then what is obtained is called the *associate surface graph* of G, and denoted by $\mathcal{H}(G)$. From Theorem 12.3, it is also called the *surface embedding graph* of G.

Theorem 12.5 In $\mathcal{H}(G)$, there is a Hamilton path. Further, for any two vertices, $\mathcal{H}(G)$ has a Hamilton path with the two vertices as ends.

Proof By arranging an order, an Hamiltonian path can be extracted based on the procedure of the layer division. □

First, starting from a surface in $\mathcal{A}(G)$, by doing exchangers at each principle in one layer to another, a Hamilton path can always be found in considering Theorem 12.3. This implies the first statement.

Further, for chosen $S_1, S_2 \in \mathcal{A}(G) = V(\mathcal{H}(G))$ adjective, starting from from S_1, by doing exchangers avoid S_2 except the final step, on the basis of the strongly finite recursion principle, a Hamilton path between S_1 and S_2 can be obtained. This implies that $\mathcal{H}(G)$ has a Hamilton circuit and hence the last statement.

This theorem tells us that the problem of determining the minimum, or maximum genus of graph G has an algorithm in time linear on $\mathcal{H}(G)$.

XII.3 Handle polynomials

Let $\mathcal{S}(G)$ be the set of associate surfaces of a graph G and $\mathcal{S}_g(G)$, the subset of $\mathcal{S}(G)$ with genus g. The the enumerating function

$$\gamma(G; z) = \sum_{g=g_{\min}}^{g_{\max}} |\mathcal{S}_g(G)| z^g \tag{12.4}$$

is called the *genus polynomial* of G where $g_{\min}$ and $g_{\max}$ are, respectively, the minimum and maximum genus of G for orientable, or nonorientable case. In orientable case, $\mu(G; x) = \gamma(G; x)$ is called the *handle polynomial*. In nonorientable case, $\nu(G; y) = \gamma(G; y)$ is the *crosscap polynomial*.

On the basis of the theory described in 12.1 and 12.2, (12.4) is in fact the genus distribution of embeddings of G. Because the enumerating function of super rooted maps of G is a constant times the genus polynomial $\gamma(G; z)$, for the enumeration of naps by genus it is enough only to discuss $\gamma(G; z)$.

Lemma 12.6 An orientable associate surface of a graph without two letters interlaced has a letter x such that xx^{-1} is a segment of the surface.

Proof Let $\langle x, x^{-1}\rangle$ be a segment of the surface with minimum of letters. If it does not contain only the letter x, then there is another letter y in it. Because of x and y noninterlaced, the segment $\langle y, y^{-1}\rangle$, or $\langle y^{-1}, y\rangle$, is a subsegment of $\langle x, x^{-1}\rangle$. However, it has at least one letter less than the minimum. □

Lemma 12.7 An orientable associate surface of a graph is with genus 0 if, and only if, no two letters are interlaced.

Proof On the basis of Lemma 12.6, by the finite recursion principle the lemma can soon be found. □

Theorem 12.6 If an orientable associate surface of a graph has two letters interlaced, *i.e.*, in form as $AxByCx^{-1}Dy^{-1}E$, then its

genus is k, $k \geq 1$, if, and only if, the orientable genus of $ADCBE$ is $k-1$.

Proof On the basis of Relation 1 in I.2, the theorem is soon found. □

According this theorem, a linear time algorithm can be designed for classifying the orientable associate surfaces of a graph G by their genus. Let $N_i(G)$ be the number of orientable associate surfaces of G with genus i, $i \geq 0$

Theorem 12.7 The handle polynomial of G is

$$\mu(G;x) = \sum_{0 \leq i \leq \lfloor \frac{\beta}{2} \rfloor} N_i(G)x^i \tag{12.5}$$

where β is the Betti number of G.

Proof From (12.4), the theorem follows. □

XII.4 Crosscap polynomials

Let $\mathcal{F}^i_{2\beta} = \{S^i_{2\beta,j} | 1 \leq j \leq s_i\}$ where s_i is the number of orientable 2β-gons (surfaces) with genus i, then

$$\mathcal{F}_{2\beta} = \sum_{0 \leq i \leq s_i} \mathcal{F}^i_{2\beta}. \tag{12.6}$$

Given a surface $S \in \mathcal{F}_{2\beta}$, S induces $2^{2\beta}-1$ nonorientable surfaces. Let $\mathcal{N}_S$ be the set of all nonorientable surfaces induced by S. then the polynomial

$$\delta_S(y) = \sum_{1 \leq j \leq \beta} |\mathcal{N}_j(S)| y^j \tag{12.7}$$

is called the *nonorientable form* of S where $\mathcal{N}_j(S)$ is the subset of $\mathcal{N}_S$, $1 \leq j \leq \beta$.

For a graph G with Betti number β, the set of all associate orientable surfaces of determined by joint trees of G is denoted by $\mathcal{S}(G)$.

Let $\mathcal{S}_\delta(G)$ for $\delta \in \Delta_{\mathcal{S}}$ be the subset of $\mathcal{S}(G)$ with nonorientable form δ where $\Delta_{\mathcal{S}}$ is the set of all nonorientable forms of surfaces in $\mathcal{S}(G)$.

Theorem 12.8 The crosscap polynomial of a graph G is

$$\nu(G;y) = \sum_{\delta\in\Delta_{\mathcal{S}}} |\mathcal{S}_\delta(G)|\delta(y). \tag{12.8}$$

Proof From (12.4) and (12.7), the theorem is deduced. □

Theorem 12.9 If a nonorientable associate surface of a graph is in form as $AxBxC$, then its genus is k, $k \geq 1$, if, and only if, the genus of $AB^{-1}C$ is

$$\begin{cases} k-1, & \text{if } AB^{-1}C \text{ is nonorientable;} \\ \dfrac{k-1}{2}, & \text{otherwise.} \end{cases} \tag{12.9}$$

Proof From Relation 2 in I.2, the theorem is soon found. □

According to Theorem 12.9, a linear time algorithm can also be designed for determining the genus of a surface and then classify nonorientable associate surfaces of a graph by genus. Hence, the crosscap polynomial expressed by (12.8) can soon be found.

Activities on Chapter XII

XII.5 Observations

O12.1 For the bouquet of size 3, list all its associate surfaces and then classify them by genus, orientable and nonorientable.

O12.2 For K_4, list all of its associate surfaces and then classify them by genus, orientable and nonorientable.

O12.3 Compare the two sets of associate surfaces obtained in O12.1 and O12.2.

O12.4 Observe how many layer divisions of the surface

$$(aa^{-1}bb^{-1}cc^{-1}).$$

O12.5 List all orientable surfaces of 4-gons.

O12.6 For each orientable surface obtained in O12.5, find its nonorientable form.

O12.7 List all orientable surfaces of 6-gons.

O12.8 For each orientable surface obtained in O12.7, find its nonorientable form.

O12.9 Find the nonorientable form of surface $(aa^{-1}bb^{-1}cc^{-1})$.

O12.10 Find the nonorientable form of surface $(abca^{-1}b^{-1}c^{-1})$.

O12.11 Find the nonorientable form of surface $(abcc^{-1}b^{-1}a^{-1})$.

XII.6 Exercises

A graph is called a *necklace* if it is Hamiltonian and the result of deleting all 2-edges is a perfect matching.

E12.1 Find the handle polynomial of a necklace with order $2n$, $n \geq 2$.

E12.2 Find the crosscap polynomial of a necklace with order $2n$, $n \geq 2$.

E12.3 Show that the nonorientable form of surface

$$(a_1 b_1 \cdots a_n b_n a_1^{-1} b_1^{-1} \cdots a_n^{-1} b_n^{-1})$$

for $n \geq 1$ is

$$x((1+x)^{2n} - x^{2n}).$$

E12.4 Show that the nonorientable form of surface

$$(a_1 a_2 \cdots a_n a_1^{-1} a_2^{-1} \cdots a_n^{-1})$$

for $n \geq 1$ is

$$x((1+x)^n - x^n).$$

E12.5 Show that the nonorientable form of surface

$$(a_1 a_1^{-1} a_2 a_2^{-1} \cdots a_n a_n^{-1})$$

for $n \geq 4$ is

$$(1+x)^n - 1.$$

E12.6 For all surfaces of 2β-gons, $\beta \geq 4$, with genus 0, Show that their nonorientable forms are

$$(1+x)^\beta - 1.$$

A graph is called a *ladder* if it has a Hamiltonian circuit and all edges not on the circuit are geometrically parallel.

E12.7 Find the handle polynomial of a ladder with m edges not on the Hamiltonian circuit.

E12.8 Find the crosscap polynomial of a ladder with m edges not on the Hamiltonian circuit.

A graph is called a *Ringel ladder* if it is cubic without multi-edge and consists of a ladder with the two 2-edges each of which is subdivided into 3 edges by two vertices and the other two edges are interlaced on the hamiltonian circuit

E12.9 Find the handle polynomial of a ladder with m edges not on the Hamiltonian circuit.

E12.10 Find the crosscap polynomial of a ladder with m edges not on the Hamiltonian circuit.

XII.7 Researches

R12.1 Find the handle polynomial of the bouquet B_n of size m, $m \geq 1$ by joint tree model.

R12.2 Find the crosscap polynomial of the bouquet of size m, $m \geq 1$ by joint tree model.

R12.3 Find the handle polynomial of the wheel W_n of order n, $n \geq 4$ by joint tree model.

R12.4 Find the crosscap polynomial of the wheel W_n of order n, $n \geq 4$ by joint tree model.

R12.5 Find the handle polynomial of the complete graph K_n of order n, $n \geq 4$ by joint tree model.

R12.6 Find the crosscap polynomial of the complete graph K_n of order n, $n \geq 4$ by joint tree model.

R12.7 Find the handle polynomial of the complete bipartite graph $K_{m,n}$ of order $m + n$, $m, n \geq 3$ by joint tree model.

R12.8 Find the crosscap polynomial of the complete bipartite graph $K_{m,n}$ of order $m + n$, $m, n \geq 3$ by joint tree model.

R12.9 Find the handle polynomial of the n-cube of order n,

$n \geq 3$ by joint tree model.

R12.10 Find the crosscap polynomial of the n-cube of order n, $n \geq 3$ by joint tree model.

R12.11 For the n-cube Q_n, $n \geq 3$, prove that the minimum genus $\gamma_n = g_{\min}(Q_n)$ with γ_{n-1} satisfies the relation

$$g_{\min}(Q_n) = 2^{n-4}(n-3) + g_{\min}(Q_{n-1})$$

from an associate surface of Q_{n-1} with genus γ_{n-1} to get an associate surface of Q_n with genus γ_n.

R12.12 For the complete bipartite graph $K_{m,n}$, $m \geq n \geq 4$, prove that the minimum genus $\gamma_{m,n} = g_{\min}(K_{m,n})$ with $\gamma_{m,n-1}$ satisfies the relation

$$g_{\min}(K_{m,n}) = \langle \frac{m-2}{4} \rangle + g_{\min}(K_{m,n-1}) - 1$$

where

$$\langle \frac{m-2}{4} \rangle = \begin{cases} \lceil \frac{m-2}{4} \rceil, \; m = 0(2 \not| n), 1(2 \not| n; 2|n, 2|\lfloor n/2 \rfloor), \\ \qquad 3(2 \not| n, 2 \not| \lfloor n/2 \rfloor; 2|n, 2|\lfloor n/2 \rfloor); \\ \frac{m-2}{4}, \; m = 2(\mathrm{mod}\ 4); \\ \lfloor \frac{m-2}{4} \rfloor, \; m = 0(2|n), 1(2|n, 2 \not| \lfloor n/2 \rfloor), \\ \qquad 3(2 \not| n, 2|\lfloor n/2 \rfloor; 2|n, 2 \not| \lfloor n/2 \rfloor) \end{cases}$$

from an associate surface of $K_{m,n-1}$ with genus $\gamma_{m,n-1}$ to get an associate surface of $K_{m,n}$ with genus $\gamma_{m,n}$.

R12.13 For the complete graph K_n, $n \geq 5$, prove that the minimum genus $\gamma_n = g_{\min}(K_n)$ with γ_{n-1} satisfies the relation

$$g_{\min}(K_n) = \langle \frac{n-4}{6} \rangle + g_{\min}(K_{n-1})$$

where

$$\langle\frac{n-4}{6}\rangle = \begin{cases} \lceil\frac{n-4}{6}\rceil, \ n = 2, 1(2 \not| \lfloor n/6\rfloor), 3(2|\lfloor n/6\rfloor), 5(2 \not| \lfloor n/6\rfloor); \\ \frac{n-4}{6}, \ n = 4(\mathrm{mod}\ 6); \\ \lfloor\frac{n-4}{6}\rfloor, \ n = 0, 1(2|\lfloor n/6\rfloor), 3(2 \not| \lfloor n/6\rfloor), 5(2|\lfloor n/6\rfloor) \end{cases}$$

from an associate surface of K_{n-1} with genus γ_{n-1} to get an associate surface of K_n with genus γ_n.

Chapter XIII

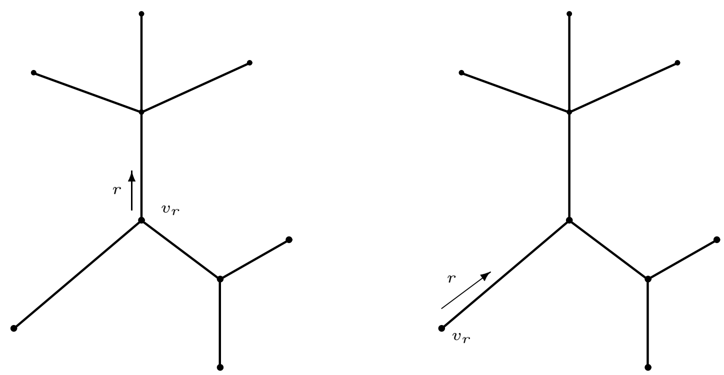
Census with Partitions

- The planted trees are enumerated with vertex partition vector in an elementary way instead as those methods used before.
- A summation free form of the number of outerplanar rooted maps is derived from the result on planted trees.
- On the basis of the result for planted outerplanar maps, the numbers of Hamilton cubic rooted maps is determined.
- The number of Halin rooted maps with vertex partition is gotten as a form without summation.
- Biboundary inner rooted maps on the sphere are counted by, an explicit formula with vertex partitions.
- On the basis of joint tree model, the number of general rooted maps with vertex partition can also expressed via planted trees in an indirected way.
- The pan-flowers which have pan-Halin maps as a special case are classified according to vertex partition and genus given.

XIII.1 Planted trees

A *plane tree* is such a super planar rooted map of a tree. A *planted tree* is a plane tree of root-vertex valency 1. In Fig.13.1, (a) shows a plane tree and (b), a planted tree.

Let T be a planted tree of order n with vertices v_0, v_1, v_2, $\cdots$, v_n, $n \geq 1$, where v_0 is the rooted vertex. The segment recorded as travelling along the face boundary of T from v_0 bach to itself and then v_0 left off is called a *V-code* of T when v_i is replaced by i for $i = 1, 2, \cdots, n$ as shown in Fig.13.2 and Fig.13.3.



(a) A plane tree (b) A planted tree

Fig.13.1 Plane tree and planted tree

A sequence of numbers is said to be *polyhedral* if each adjacent pair of numbers occurs twice. It is easily seen that a V-code of a planted tree is a polyhedral segment.

The vector $\underline{n} = (n_1, n_2, \cdots, n_i, \cdots)$, where $n_i (i \geq 1)$ is the number of unrooted vertices of valency i, is called the *vertex partition* of a planted tree.

For a sequence of nonnegative integers n_1, n_2, $\cdots$, n_i, $\cdots$ denoted by a vector $\underline{n} = (n_1, n_2, \cdots, n_i, \cdots)$, if

$$\sum_{i \geq 1} (2 - i) n_i = 1, \tag{13.1}$$

then $\underline{n}$ is said to be *feasible*. Let

$$\underline{n}' = (n_1', n_2', \cdots, n_i', \cdots)$$

where

$$n'_i = \begin{cases} n_1 - 1, \text{ when } i = 1; \\ n_{k-1} + 1, \text{ when } i = k-1; \\ n_k - 1, \text{ when } i = k; \\ n_i, \text{ otherwise} \end{cases} \tag{13.2}$$

for a $k \geq 2$ and $n_1, n_k > 0$, then $\underline{n}'$ is called a *reduction* of $\underline{n}$.

Lemma 13.1 A reduction $\underline{n}'$ of a sequence of nonnegative integers $\underline{n}$ is feasible if, and only if $\underline{n}$ is feasible.

Proof By considering (13.2), we have the eqaulity as

$$\sum_{i\geq 1}(2-i)n'_i - \sum_{i\geq 1}(2-i)n_i = -1 - (2-k) + (2-k+1) = 0.$$

This leads to the lemma. □

The sequence $\underline{n}_0 = (1, 1)$ is feasible but no reduction can be done. So, it is called *irreducible*.

Lemma 13.2 Any feasible sequence $\underline{n}$ has $n_1 > 0$.

Proof By contradiction. Suppose $\underline{n}$ is feasible but $n_1 = 0$. Because of

$$\sum_{i\geq 1}(2-i)n_i = \sum_{i\geq 2}(2-i)n_i \leq 0.$$

This contradicts to (13.1), the feasibility. □

Lemma 13.3 Any feasible sequence $\underline{n} \neq \underline{n}_0$ can always be transformed into $\underline{n}_0$ only by reductions.

Proof Because of $\underline{n} \neq \underline{n}_0$, Lemma 13.2 enables us to get a reduction. Whenever the reduction is not $\underline{n}_0$, another reduction can also be done from Lemma 13.1. By the finite recursion principle, the lemma is done. □

Theorem 13.1 For a nonnegative integer sequence $\underline{n} = (n_1, n_2, \cdots, n_i, \cdots)$, there exists such a planted tree that n_i unrooted vertices are of valency $i (i \geq 1)$ if, and only if, $\underline{n}$ is feasible.

Proof Necessity. Suppose T is such a planted tree. Because n_i is the number of unrooted vertices with valency i, $i \geq 1$ in T, the size of T is

$$\sum_{i\geq 1} n_i$$

and hence

$$1+\sum_{i\geq 1} in_i = 2\sum_{i\geq 1} n_i.$$

This means that $\underline{n}$ satisfies (13.1), *i.e.*, $\underline{n}$ is feasible.

Sufficiency. First, it is seen that the irreducible sequence is the vertex partition of the planted tree whose under graph is a path of two edges. Then, by following the inversion of the procedure in the proof of Lemma 13.3, a planted tree with a given feasible sequence can be found. $\square$

For a polyhedral segment L with 1 as both starting and ending numbers on the set $N = \{1, 2, 3, \cdots, n\}$, $n \geq 1$, let the vector be the *point partition* of L where n_i be the number of occurrences of i in L, $i \geq 1$.

In a polyhedral segment L, if vuv is a subsegment of L, then u is said to be *contractible*. The operation of deleting u and then identifying v, or in other words vuv is replaced by v, is called *contraction*. If L can be transformed into a single point, then L is called a *celluliform*.

If the point partition of L satisfies (13.1), then L is said to be *feasible* as well.

It can be seen that any celluliform is a feasible segment but conversely not necessary to be true.

In what follows, the notation bellow is adopted as

$$\binom{n}{\underline{n}} = \binom{n}{n_1, n_2, \cdots, n_s} = \prod_{i=1}^{s-2} \binom{n-\sigma_{i-1}}{n_i} \tag{13.3}$$

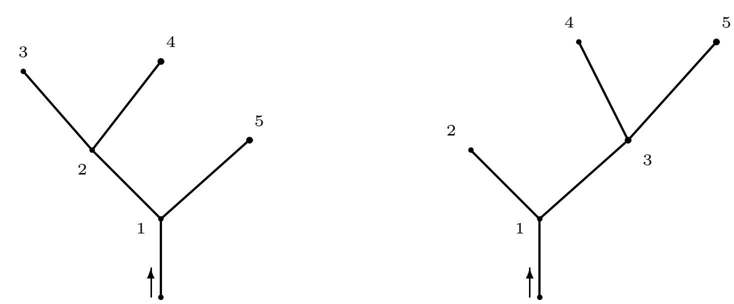
where $s \geq 2$, $n_i \geq 0$ are all integers and

$$n = \sum_{i=1}^{s} n_i,\ \ \sigma_{i-1} = \sum_{j=1}^{i-1} n_j.$$

Notice that when $s = 2$, it becomes the combination of choosing n_1 from n.

Example 13.1 In Fig.13.2, two distinct planted trees of order 5 are with vertex partition $\underline{n} = (3, 0, 2)$ satisfying (13.1). (a) is with sequence 123242151 and (b), 121343531. Here,

$$\frac{1}{5}\binom{5}{3,0,2} = \frac{1}{5}\frac{5!}{3!0!2!} = 2.$$



(a) 123242151 (b) 121343531

Fig.13.2 Trees with $\underline{n} = (3, 0, 2)$

In Fig.13.3, six distinct planted trees of order 5 shown by (a–f) are with vertex partition $\underline{n} = (2, 2, 1)$ satisfying (13.1). Here,

$$\frac{1}{5}\binom{5}{2,2,1} = \frac{1}{5}\frac{5!}{2!2!1!} = 6.$$

For a feasible segment of numbers on N, the occurrences of $i \in N$ divides the segment into sections in number equal to that of times of its occurrences. Each of the sections is called an *i-section*.

If a feasible segment on N is with the property that all numbers less than i have occurred before the first occurrence of i, $1 \leq i \leq n$, then it is called *favorable*. Denote by $1\Leftrightarrow 1$ a 1-to-1 correspondence between two sets.

Lemma 13.4 Let $\mathcal{T}_{\underline{n}}$ be the set of all planted trees of order $n+1$ with vertex partition $\underline{n}$ and $\mathcal{L}_{\underline{n}}$, the set of all favorable celluliforms on N with point partition $\underline{n}$, then $\mathcal{T}_{\underline{n}}\ 1\Leftrightarrow 1\ \mathcal{L}_{\underline{n}}$.

Proof Necessity. For $T \in \mathcal{T}_{\underline{n}}$, it is easy to check that its V-code $\mu(T)$ is uniquely a favorable celluliform, *i.e.*, $\mu(T) \in \mathcal{L}_{\underline{n}}$.

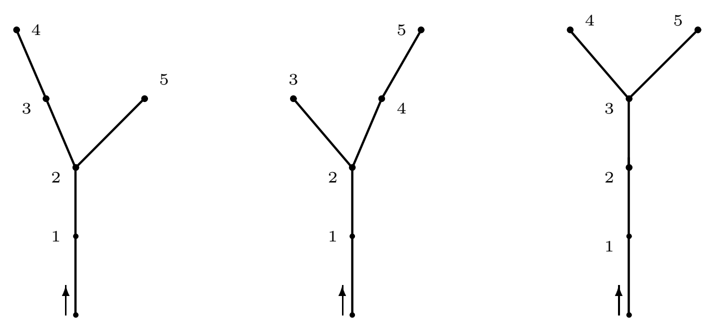
Sufficiency. Let $\mu \in \mathcal{L}_{\underline{n}}$. Because of the uniqueness of the greatest point which is contractible, a point can be done by successfully contracting the greatest points. By reversing the procedure, a tree $T(\mu) \in \mathcal{T}_{\underline{n}}$ is done. $\square$

Theorem 13.2 The number of nonisomorphic planted trees of order $n+1$ with vertex partition $\underline{n}$ is

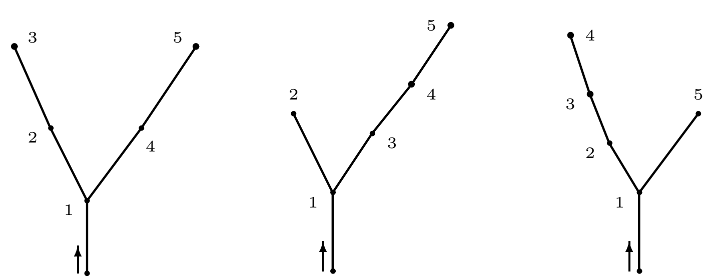
$$\frac{1}{n}\binom{n}{\underline{n}} = \frac{(n-1)!}{\underline{n}!} \tag{13.4}$$

where

$$\underline{n}! = \prod_{i \geq 1} n_i!. \tag{13.5}$$



(a) 123432521 (b) 1232345421 (c) 123435321



(d) 123214541 (e) 121345431 (f) 123432151

Fig.13.3 Trees with $\underline{n} = (2, 2, 1)$

Proof On the basis of Lemma 13.4, it suffices to discuss the set of all favorable celluliforms $\mathcal{L}_{\underline{n}}$. Since each favorable celluliform has n possibilities to choose the minimum point and different possibilities correspond to different ways of choosing $\underline{n}$ from n elements, the set of all ways is partitioned into

$$\frac{1}{n}\binom{n}{\underline{n}}$$

classes. A way is represented by number sequence of length n with repetition as occurrence in the natural order. Two ways A and B are equivalent if, and only if, there exists a number $i \in N$ such that $A + i(\mathrm{mod}\ n)$ is B in cyclic order. A way starting from 1 is said to be *standard*. Because of each class with n ways in which only the standard way enables us to form the V-code of a planted tree, the theorem is soon obtained. □

In Example 13.1, Fig.13.2 and Fig.13.3 show two cases of (13.4). Only take $\underline{n} = (3, 0, 2)$ as an example. There are 10 ways of combinations of choosing 2 points with 3 occurrences each and 3 points with 1 occurrence each from 5 points numbered by $1, 2, 3, 4$ and 5 as

(1) 111222345; (2) 111233345; (3) 111234445;
(4) 111234555; (5) 122233345; (6) 122234445;
(7) 122234555; (8) 123334445; (9) 123334555;
(10) 123444555

in which 2 classes are divided as $C_1 = \{(1), (5), (8), (10), (4)\}$ and $C_2 = \{(2), (6), (9), (3), (7)\}$ because of $(5) = 222333451$, $(8) = 333444512$, $(10) = 444555123$ and $(4) = 555111234$ as $(1) = 111222345$ for C_1, and the like for C_2.

For a general outerplanar rooted map $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$ with $(r)_{\mathcal{P}\gamma}$ on the specific circuit where r is the root and $\gamma = \alpha\beta$ and its dual $M^* = (\mathcal{X}_{\beta,\alpha}(X), \mathcal{P}\gamma)$ with root r as well without loss of generality, let H_M be the map obtained from M^* by transforming the vertex $(r)_{\mathcal{P}\gamma}$ of M^* into vertices (r), $((\mathcal{P}\gamma)r)$, $\cdots$, $((\mathcal{P}\gamma)^{-1}r)$. Such an operation is called *articulation*. The root r_H of H_M is taken r as shown in

Fig.13.4 in which bold lines are on M and dashed lines, on H_M. Here, multiedges are permitted.

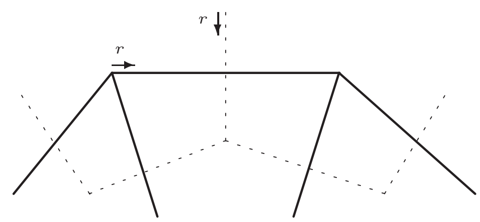


Fig.13.4 M and H_M

Further, it is easily checked that H_M is a planted tree of size which is equal to the size of M.

Lemma 13.5 An outerplanar rooted map M of order $n+1$ with face partition $\underline{s}$ is 1–to–1 corresponding to a planted tree H_M with vertex partition $\underline{t} = \underline{s} + n\underline{1}_1$.

Proof By the procedure of getting H_M from M, it is seen that the number of i-faces in M is the same as that in H_M for $i > 1$.

For $i = 1$, H_M has $n-1$ articulate vertices greater than s_1. In virtue of the nonseparability of M, $s_1 = 0$.

Conversely, it is still true and hence the lemma. □

An attention which should be paid to is that all articulate edges in H_M are 1–to–1 corresponding to all edges on the root-face boundary of M.

Theorem 13.3 The number of nonisomorphic outerplanar rooted maps of order n with face partition $\underline{s}$ is

$$\frac{1}{n+s-1}\binom{n+s-1}{\underline{s}+(n-1)\underline{1}_1} \tag{13.6}$$

where $s_1 + n = n$, *i.e.*, $s_1 = 0$, because of no articulate vertex and s is the number of unrooted faces.

Proof On the basis of Lemma 13.5, the theorem is obtained from Theorem 13.2. □

Since a bipartite map has all of its faces of even valency, its face partition $\underline{s}$ is of all $s_i = 0$ when i is even.

Corollary 13.1 The number of nonisomorphic outerplanar rooted bipartite maps of order $2m$ with face partition $\underline{s}$ is

$$\frac{1}{2m+s-1}\binom{n+s-1}{\underline{s}+(2m-1)\underline{1}_1} \tag{13.7}$$

where $s_1 + n = n$, *i.e.*, $s_1 = 0$, because of no articulate vertex and s is the number of unrooted faces. □

A map is said to be *simple* if it has neither selfloop nor multiedge.

Corollary 13.2 The number of nonisomorphic outerplanar rooted simple maps of order n with face partition $\underline{s}$ is

$$\frac{1}{n+s-1}\binom{n+s-1}{\underline{s}+(n-1)\underline{1}_1} \tag{13.8}$$

where $s_1 + n = n$, *i.e.*, $s_1 = 0$, because of no articulate vertex and s is the number of unrooted faces. □

Corollary 13.3 The number of nonisomorphic outerplanar rooted bipartite maps of order $2m$ with face partition $\underline{s}$ is

$$\frac{1}{2m+s-1}\binom{n+s-1}{\underline{s}+(2m-1)\underline{1}_1} \tag{13.9}$$

where $s_1 + n = n$, *i.e.*, $s_1 = 0$, because of no articulate vertex and s is the number of unrooted faces. □

XIII.2 Hamiltonian cubic map

For saving the space occupied, this section concentrate to discuss on Hamiltonian planar rooted quadregular maps as super maps of a Hamiltonian planar graph, and then provide a main idea for general

such maps. A map is said to be *quadregular* if each of its vertices is of valency 4.

A Hamiltonian planar rooted quadregular map with the two edges not on the Hamiltonian circuit not success in the rotation at each vertex is called a *quaternity*.

Let $M_1 = (\mathcal{X}_{\alpha,be}(X_1), \mathcal{J}_1)$ and $M_2 = (\mathcal{X}_{\alpha,be}(X_2), \mathcal{J}_2)$ be two rooted maps with their roots, respectively, r_1 and r_2. Assume $(r_1)_{\mathcal{J}_1\gamma}$ and $(r_2)_{\mathcal{J}_2\gamma}$ are with the same length.

The map obtained by identifying r_1 and αr_2 with $Kr_1 = K\alpha r_2$ as well as $(\mathcal{J}_1\gamma)^i r_1$ and $(\alpha\mathcal{J}_2\gamma)^i r_2$ for $i \geq 1$ is called the *boundary identification* of M_1 and M_2, denoted by $I(M_1, M_2)$. The operation from M_1 and M_2 to $I(M_1, M_2)$ is called *boundary identifier*.

A boundary identification of two outerplanar cubic rooted maps is a quaternity because of M_1 and M_2 both outerplanar and cubic with its root $r_1 = r_2$.

Lemma 13.6 Let $\mathcal{Q}_{\underline{n}}$ and $\mathcal{I}_{\underline{n}}$ be the sets of all, respectively, quaternities and boundary identifiers with face partition $\underline{s}$, then there is a 1–to–1 correspondence between $\mathcal{Q}_{\underline{s}}$ and $\mathcal{I}_{\underline{s}}$.

Proof By considering the inverse of a boundary identifier, a quaternity becomes two cubic outerplanar maps whose boundary identification is just the quaternity with the same face partition $\underline{s}$. This is the lemma. □

From the proof of this lemma, it is seen the identity

$$\mathcal{Q}_{\underline{s}} = \mathcal{I}_{\underline{s}}. \tag{13.10}$$

Lemma 13.7 The number of nonisomorphic outerplanar cubic rooted maps of order n with face partition $\underline{s}$ is

$$\frac{1}{n+s-1}\binom{n+s-1}{\underline{s}+(n-1)\underline{1}_1} \tag{13.11}$$

for $\underline{s} \in \mathcal{S}_{\text{cub}}$, the set of all the vectors available as the face partition of an outerplanar cubic map.

Proof From Theorem 13.3, the conclusion is true. □

Theorem 13.4 The number of nonisomorphic quaternities of order n with face partition $\underline{s}$ is

$$\sum_{\substack{\underline{s}_1,\underline{s}_2\in\mathcal{S}_{\text{cub}}\\ \underline{s}=\underline{s}_1+\underline{s}_2}} \frac{\binom{n+a_1-1}{\underline{s}_1+(n-1)\underline{1}_1}\binom{n+a_2-1}{\underline{s}_2+(n-1)\underline{1}_1}}{(n+a_1-1)(n+a_2-1)} \tag{13.12}$$

where $a_i = |\underline{s}_i|$, called the *absolute norm* of $\underline{s}_i$, *i.e.*, the sum of all the absolute values of entries in $\underline{s}_i$ for $i = 1, 2$.

Proof Since the set of all quaternities of order n is the Cartesian product of the set of all cubic outerplanar rooted maps and itself, the formula (13.12) is soon obtained. □

This method can be also employed for the case when the boundary is cubic and further for others with observing boundary combinatorics.

XIII.3 Halin maps

If a graph can be partitioned into a tree and a circuit whose vertex set consists of all articulate vertices of the tree, then it is called a *Halin graph*. A planar *Halin map* is a super map of a Halin graph on the surface of genus 0 such that the circuit forms a face boundary.

Let $H = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{J})$ be a planar Halin rooted map with $(r)_{\mathcal{J}\gamma}$, $\gamma = \alpha\beta$, as the face formed by the specific circuit where r is the root. The associate planted tree denoted by T_H is obtained by deleting all the edges Kr, $K(\mathcal{J}\gamma)r$, $\cdots$, $K(\mathcal{J}\gamma)^{-1}r$ on the circuit.

Lemma 13.8 A planar Halin rooted map with vertex partition $\underline{u}$ of the specific circuit with length n is 1–to–1 corresponding to a planted tree with vertex partition $\underline{v} = \underline{u} + (n-1)(\underline{1}_1 - \underline{1}_3)$.

Proof By considering the procedure from a Halin map H to a tree

T_H, carefully counting the numbers of vertices with the same valency and comparing them of H with those of T_H, the lemma is found. □

Theorem 13.5 The number of nonisomorphic planar Halin rooted map with vertex partition $\underline{u}$ of the specific circuit with length n is

$$A(\mathcal{H}_n^{\underline{u}}) = \frac{1}{|\underline{u}|+n-1}\binom{|\underline{u}|+n-1}{\underline{u}+(n-1)(\underline{1}_1-\underline{1}_3)} \tag{13.13}$$

where $|\underline{u}|$ is the absolute norm of $\underline{u}$.

Proof On the basis of Lemma 13.8, by Theorem 13.2, the conclusion of the theorem is done. □

Let $H_1 = (\mathcal{X}_{\alpha,\beta}(X_1), \mathcal{J}_1)$ and $H_2 = (\mathcal{X}_{\alpha,\beta}(X_2), \mathcal{J}_2)$ be two planar Halin rooted maps with $|\{r_1\}_{\mathcal{J}_1\gamma}| = |\{r_2\}_{\mathcal{J}_2\gamma}|$, the boundary identification of H_1 and H_2 is called a *double leaf*.

A graph with a specific circuit of all vertices of valency 4 is called a *quadcircularity*. A super map of a quadcircularity is a *quadcirculation*.

Lemma 13.9 A planar rooted quadcirculation M is a double leaf if, and only if, the map obtained from M by deleting all edges on the specific circuit can be partitioned into two trees such that each of vertices on the circuit is articulate of both the trees.

Proof Since a double leaf is obtained by boundary identifier from two Halin maps, the conclusion is directly deduced. □

Lemma 13.10 A planar rooted quadcirculation with vertex partition $\underline{u}$ of the specific circuit of length n is 1–to–1 corresponding to a pair of planar Halin rooted maps H_1 and H_2 with vertex partitions, respectively, $\underline{s}$ and $\underline{t}$ such that

$$\underline{u} = \underline{s} + \underline{t} - (n-1)(2\underline{1}_3 - \underline{1}_4). \tag{13.14}$$

where $\underline{1}_i$ is the vector of all entries 0 but the i-th 1 for $i = 3, 4$.

Proof By considering that $\underline{u}$ does not involve $n-1$ unrooted

3-vertices in $\underline{s}$ and $\underline{t}$ each and involves $n-1$ unrooted 4-vertices, the formula (13.14) holds. □

Theorem 13.6 The number of nonisomorphic double leafs with vertex partition $\underline{u}$ of the specific circuit of valency n is

$$\sum_{\substack{\underline{s}+\underline{t}=\underline{u}-(n-1)(2\underline{1}_3-\underline{1}_4)\\ \underline{s},\underline{t}\in\mathcal{S}_{\mathrm{dl}}}} A(\mathcal{H}_n^{\underline{s}})A(\mathcal{H}_n^{\underline{t}}) \tag{13.15}$$

where $\mathcal{S}_{\mathrm{dl}}$ is the set of vectors available as vertex partitions of planar Halin maps.

Proof On account of Lemma 13.10, the theorem is soon derived from Theorem 13.2 for the Cartesian product of two sets. □

Given a nonseparable graph G with a cocircuit C^* of an orientation defined, if G is planar in companion with such a orientation then G is said to have the *C^*-oriented planarity*, or *cocircuit oriented planarity*. A planar super map of such a graph is called a *cocircuit oriented map*. If each edge on the cocircuit is bisectioned and then snip off each new 2-valent vertex as two articulate vertices in a cocircuit oriented map M so that what obtained is two disjoint plane trees, then M is called a *cocircular map*. The root is always chosen to be an element in an edge on the cocircuit in a cocircular map.

Lemma 13.11 A cocircular map with the oriented cocircuit of $n+1$ edges and the vertex partition $\underline{u}$ is 1–to–1 corresponding to a pair of planted trees $\langle T_1, T_2\rangle$ with vertex partitions $\underline{u}_1$ and $\underline{u}_2$ such that $u_{11} = u_{21} = n$.

Proof By considering the uniqueness of a cocircular map composed from two planted trees, the coclusion is directly deduced. □

Let $\mathcal{U}_n$ be the set of all integer vectors feasible to a planted tree with n unrooted articulate vertices.

Theorem 13.8 The number of nonisomorphic cocircular maps

with the oriented cocircuit of n edges and given vertex partition $\underline{u}$ is

$$\sum_{\substack{\underline{u}_1,\underline{u}_2\in\mathcal{U}_n\\ \underline{u}_1+\underline{u}_2=\underline{u}}} \frac{\binom{|\underline{u}_1|}{\underline{u}_1}\binom{|\underline{u}_2|}{\underline{u}_2}}{|\underline{u}_1||\underline{u}_2|} \tag{13.16}$$

Proof Based on Lemma 13.11, the formula (13.16) is derived from Theorem 13.2. □

A *cocirculation* is such a planar rooted map which has a cocircuit oriented. For this type of planar maps, the number of nonsepsrable ones can be determined from maps with cubic boundary of root-face.

More interestingly, maps with cubic boundary of root-face can be transformed into maps with root-vertex valency as a parameter.

In view of this, many types of planar maps with cubic boundary can be known from what have been done for counting maps with size and root-vertex valency as two parameters.

XIII.4 Biboundary inner rooted maps

A map is said to be *biboundary* if it has a circuit C that two trees are obtained by deleting all the edges on C. In view of this, a Hamilton cubic map is a *uniboundary* map because it is not necessary to have two connected components as all the edges on the Hamiltonian circuit are deleted. Here, only planar case is considered.

Let$M = (\mathcal{X}, \mathcal{J})$be a biboundary map, $r = r(M)$ is its root. The length of boundary is m and the vertex partition vector of nonboundary vertices is $\underline{n} = (n_1, n_2, \cdots)$, n_i, $i \geq 1$, the number of i-vertices not on the boundary.

Assume $M_1 = (\mathcal{X}_1, \mathcal{J}_1)$ and $M_2 = (\mathcal{X}_2, \mathcal{J}_2)$ are two submaps of M. Denote by$C = (Kr, K\varphi r, \cdots, K\varphi^{m-1}r)$ the boundary circuit of M where

$$\varphi x_i = \begin{cases} \mathcal{J}\gamma x_i, & \text{if } \gamma x_i \text{ not incident with an inner edge of } M_2; \\ \alpha\mathcal{J}\beta x_i, & \text{otherwise.} \end{cases}$$

$x_0 = x_m = r$ and $x_i = \varphi^i r$, $i = 1, 2, \cdots, m-1$.

Let $r_1 = r(M_1) = r$ and $r_2 = r(M_2) = \alpha\varphi^{m-1}r$. This means the root-vertex of M_1 adjacent to that of M_2 in M. Such a root is said to be *inner rooted*.

First, denoted by $\mathcal{B}_m$, $m \geq 6$, the set of all biboundary rooted maps with the boundary length$m$.

Lemma 13.12 Let $\mathcal{W}_{m_1}$ and $\mathcal{W}_{m_1}$ are, respectively, uniboundary maps of boundary lengths $m_1 \geq 3$ and $m_2 \geq 3$, then a pair of $\{W_1, W_2\}$, $W_1 \in \mathcal{W}_{m_1}$ and $W_2 \in \mathcal{W}_{m_2}$, $m = m_1 + m_2$, composes of

$$s_{m_2}(m_1) = \begin{bmatrix} m_2 \\ m_1 \end{bmatrix} \tag{13.17}$$

biboundary maps in $\mathcal{B}_m$. And, this combinatorial number is determined by the recursion as

$$\begin{cases} \begin{bmatrix} m_2 \\ m_1 \end{bmatrix} = \sum\limits_{i=0}^{m_1} \begin{bmatrix} m_2 - 1 \\ i \end{bmatrix}, \ m_2 \geq 2 \\ \\ \begin{bmatrix} m_2 \\ 0 \end{bmatrix} = 1, \ m_2 \geq 1; \ \begin{bmatrix} 1 \\ m_1 \end{bmatrix} = 1, \ m_1 \geq 0. \end{cases} \tag{13.18}$$

Proof By induction on m_2, $m_2 \geq 2$, for any $m_1 \geq 1$.

First, check the case of $m_2 = 2$, for $m_1 \geq 1$, that

$$\begin{bmatrix} 2 \\ m_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \cdots + \begin{bmatrix} 2 \\ m_1 \end{bmatrix} = m_1 + 1.$$

From the fact that if o vertices in the first segment, then the second segment has to have m_1 vertices; if 1 vertex in the first segment, then the second segment has to have $m_1 - 1$ vertices; $\cdots$; if m_1 vertices in the first segment, then the second segment has to have 0 verticesm_1. They are all together $m_1 + 1$. Thus, (13.18) is true for $m_2 = 2$.

Then, assume $t_{m_2-1}(m_1)$, $m_2 \geq 3$, have been determined by (13.18). To prove $t_{m_2}(m_1)$ is determined by (13.18). Because of $m_1 + 1$ occurrences for putting m_1 vertices in to m_2 segments as

when m_1 vertices in the first segment, then no vertex in all other $m_2 - 1$ segments and hence $s_{m_2-1}(0)$ ways; when $m_1 - 1$ vertices in the first segment, then i vertex in all other $m_2 - 1$ segments and hence $s_{m_2-1}(1)$ ways; $\cdots$; when 0 vertices in the first segment, then m_1 vertices in all other $m_2 - 1$ segments and hence $s_{m_2-1}(m_1)$ ways. They are $s_{m_2}(m_1) = \sum_{i=0}^{m_1} s_{m_2-1}(i)$ ways all together. That is

$$\left[\begin{array}{c} m_2 \\ m_1 \end{array}\right] = \sum_{i=0}^{m_1} \left[\begin{array}{c} m_2 - 1 \\ i \end{array}\right], \ m_2 \geq 2.$$

By the induction hypothesis, $s_{m_2}(m_1)$ is determined. The lemma is true. □

Then, denote by $\mathcal{D}_m$, $m \geq 6$, the set of all biboundary rooted maps with the boundary length m.

Lemma 13.13 Let $\mathcal{M}_{m_1}$ and $\mathcal{M}_{m_1}$ be the uniboundary rooted maps of boundary lengths, respectively, $m_1 \geq 3$ and $m_2 \geq 3$, then a pair $\{M_1, M_2\}$, $M_1 \in \mathcal{M}_{m_1}$ and $M_2 \in \mathcal{M}_{m_2}$, composes

$$t_{m_2}(m_1) = \left\langle \begin{array}{c} m_2 \\ m_1 \end{array} \right\rangle \tag{13.19}$$

biboundary rooted maps in $\mathcal{D}_m$, $m = m_1 + m_2$. And, this combinatorial number is determined by

$$\left\langle \begin{array}{c} m_2 \\ m_1 \end{array} \right\rangle = \sum_{i=0}^{m_1-1} \left[\begin{array}{c} m_2 - 1 \\ i \end{array}\right], \tag{13.20}$$

where the terms on the right hand side of (13.19) are given in Lemma 13.12.

Proof By induction on m_2 for any $m_1 \geq 1$.

First, when $m_2 = 2$, By considering for assigning m_1 vertices in M_2 edges on the boundary in the order determined that when 1, 2, $\cdots$, m_1 vertices in the first edge(incident with the root), the second edge has to have, respectively, $m_1 - 1$, $m_1 - 2$, $\cdots$, 0 vertices, we have $t_2(m_1) = s_1(0) + s_1(1) + \cdots + s_1(m_1 - 1) = m_1$.

Then, assume $t_{m_2-1}(m_1)$, $m_2 \geq 3$, have been determined by (13.20). To prove that

$$t_{m_2}(m_1) = s_{m_2-1}(0) + s_{m_2-1}(1) + \cdots + s_{m_2-1}(m_1-1),$$

is determined by (13.20) as well. Because of the first edge in M_2 edges allowed to have m_1, $m_1 - 1$, $\cdots$, 1 vertex, the other $m_2 - 1$ edges only allowed to have, respectively, 0, 1, $\cdots$, $m_1 - 1$ vertex. This implies

$$\begin{bmatrix} m_2-1 \\ 0 \end{bmatrix} + \begin{bmatrix} m_2-1 \\ 1 \end{bmatrix} + \cdots + \begin{bmatrix} m_2-1 \\ m_1-1 \end{bmatrix}.$$

Hence, (13.20) is right. □

Denote by Q_i, $1 \leq i \leq t_{m_2}(m_1)$ the $t_{m_2}(m_1)$ biboundary inner rooted maps mentioned in this lemma with $\mathcal{M}(M_1, M_2) = \{Q_1, Q_2, \cdots, Q_{t_{m_2}(m_1)}\}$.

Lemma 13.14 Let $\mathcal{H}_m = \{\mathcal{M}(M_1, M_2) | \forall (M_1, M_2) \in \mathcal{M}_{m_1} \times \mathcal{M}_{m_2},\ m_1 + m_2 = m\}$, then

$$\mathcal{D}_m = \sum_{H \in \mathcal{H}} H, \tag{13.21}$$

i.e., $\mathcal{H}_m$ is a partition of $\mathcal{D}_m$.

Proof For any $D \in \mathcal{D}_m$, it is known from biboundary maps that there exist m_1 and m_2, $m_1 + m_2 = m$, such that $M_1 \in \mathcal{M}_{m_1}$ and $M_2 \in \mathcal{M}_{m_2}$ compose of D. Thus, $D = (M_1, M_2) \in \mathcal{H}_m$.

Conversely, for any $Q \in H$, $H \in \mathcal{H}_m$, because of Q composed from two uniboundary maps $M_1 \in \mathcal{M}_{m_1}$ and $M_2 \in \mathcal{M}_{m_2}$, $m_1 + m_2 = m$, there exist $D \in \mathcal{D}_m$ such that $D = Q$.

In summary, the lemma is obtained. □

In what follows, observe how many nonisomorphic uniboundary maps of boundary length m with vertex partition vector $\underline{n}$.

Lemma 13.15 The number of uniboundary rooted maps of boundary length m, $m \geq 3$, and nonboundary vertex partition vector

$\underline{n}$ is

$$\eta(m, \underline{n}) = \frac{(m+n-1)!}{(m-1)!\underline{n}!}, \tag{13.22}$$

where $n = |\underline{n}| = n_1 + n_2 + \cdots$.

Proof Let $M = (\mathcal{X}, \mathcal{J})$ be a uniboundary rooted maps of boundary length m, $m \geq 3$, and nonboundary vertex partition vector $\underline{n}$. Its root is r. Because of cubicness on the boundary, $\mathcal{J}r$ is incident with an articulate vertex of the tree. Let $\mathcal{J}r$ be the root to make the tree planted. Because of all 1-vertices of the planted tree on the boundary, its vertex partition vector is $\underline{n}+(m-1)\underline{1}_1$, where $\underline{1}_1$ is the vector of all entries 0 but the first entry 1. Since a planted tree with vertex partition is 1-to-1 corresponding to a uniboundary rooted map of boundary length m, $m \geq 3$, and vertex partition vector $\underline{n}$, from Theorem 13.2, the number of nonisomorphic uniboundary rooted maps f boundary length m, $m \geq 3$, and vertex partition vector $\underline{n}$ is

$$\eta(m, \underline{n}) = \frac{(m+n-1)!}{(m+n_1-1)!(\underline{n}-n_1\underline{1}_1)!},$$

where $n = |\underline{n}| = n_1 + n_2 + \cdots$. By considering that $n_1 = 0$ in $\underline{n}$, the lemma is done. $\square$

On the basis of the above tree lemmas, the main result of this section can be gotten.

Theorem 13.9 The number of biboundary inner rooted maps of boundary length $m \geq 6$ and nonboundary vertex partition is

$$\sum_{(m_1,m_2,\underline{n}^1,\underline{n}^2)\in\mathcal{L}} \left\langle \begin{matrix} m_2 \\ m_1 \end{matrix} \right\rangle \frac{(n^1+m_1-1)!}{(m_1-1)!\underline{n}^1!} \times \frac{(n^2+m_2-1)!}{(m_2-1)!\underline{n}^2!} \tag{13.23}$$

where $\mathcal{L} = \{(m_1, m_2, \underline{n}^1, \underline{n}^2) | m_1 + m_2 = m, \underline{n}^1 + \underline{n}^2 = \underline{n}, m_1, m_2 \geq 3\}$.

Proof For any given m_1 and m_2, $m_1 + m_2 = m$, with $\underline{n}^1$ and $\underline{n}^2$,

$\underline{n}^1 + \underline{n}^2 = \underline{n}$, from Lemma 13.14–15, $\mathcal{D}_m$ can be classified into

$$\sum_{(m_1,m_2,\underline{n}^1,\underline{n}^2)\in\mathcal{L}} \frac{(n^1+m_1-1)!}{(m_1-1)!\underline{n}^1!}\frac{(n^2+m_2-1)!}{(m_2-1)!\underline{n}^2!}$$

classes. From Lemma 13.13, each class has

$$\left\langle \begin{array}{c} m_2 \\ m_1 \end{array} \right\rangle$$

nonisomorphic biboundary inner rooted maps of boundary length m and nonboundary vertex partition vector $\underline{n}$. Thus, the theorem is proved. □

XIII.5 General maps

Based on the joint tree model shown in Chapter XII, it looks general maps on surfaces in a closed relation with joint trees. In this section, only orientable case is considered as an instance.

Because of the independence with a tree chosen, general maps with a cotree marked are particularly investigated.

For the convenience for description, all maps are assumed to have no articulate edge.

Let $M = (\mathcal{X}, \mathcal{J})$ be a map with cotree edges $a_1 = Kx_1$, $a_2 = Kx_2$, $\cdots$, $a_l = Kx_l$ marked where $l = \beta(M)$ is the Betti number of M. The root of M is chosen on a cotree edge, assume $r = r(M) = x_1$.

Another map $H_M = (\mathcal{X}_H, \mathcal{J}_H)$ is constructed as

$$\mathcal{X}_H = \mathcal{X} + \sum_{i=1}^{l}(Ks_i + Kt_i - a_i) \tag{13.24}$$

where $Ks_i = \{x_i, \alpha x_i, \beta s_i, \gamma s_i\}$and $Kt_i = \{\gamma x_i, \beta x_i, \beta t_i, \gamma t_i\}$, $1 \leq i \leq l$, ; $\mathcal{J}_H$ is defined as

$$(x)_{\mathcal{J}_H} = \begin{cases} (x)_{\mathcal{J}}, \text{ when } x \in \mathcal{X}; \\ (x), \text{ when } x \notin \mathcal{X}. \end{cases} \tag{13.25}$$

Lemma 13.16 For any rooted map M with a cotree marked, the map H_M is a planted tree with the number of articulate vertices two times the Betti number of M.

Proof Because of connected without circuit on H_M, H_M is a tree in its own right. Since the number of cotree edges is the Betti number of M, from the construction of H_M and no articulate edge on M, the number of articulate vertices on H_M is two times the Betti number of M. $\square$

Let $\mathcal{M}(l;\underline{n})$ be the set of all general rooted maps with a cotree of size l marked and vertex partition $\underline{n}$ including the root-vertex. And, let $\mathcal{H}(\underline{n})$ be the set of all planted trees with articulate vertices two times the number of cotree edges and vertex partition $\underline{n}$ excluding the root-vertex.

Lemma 13.17 There is a 1-to-1 correspondence between $\mathcal{M}(l;\underline{n})$ and $\mathcal{H}(\underline{n}+(2l-1)\underline{1}_1)$ as the set of joint trees.

Proof For $M \in \mathcal{M}(l;\underline{n})$, it is easily seen that the corresponding H_M is just a joint tree of M and hence $H_M \in \mathcal{H}(\underline{n}+(2l-1)\underline{1}_1)$.

Conversely, for $H \in \mathcal{H}(\underline{n}+(2l-1)\underline{1}_1)$, in virtue of a joint tree with its articulate vertices are pairwise marked as cotree edges of the corresponding map M as $H = H_M$, by counting the valencies of vertices, it is checked that $M \in \mathcal{M}(l;\underline{n})$.

Therefore, the lemma is true. $\square$

This lemma enables us to determine the number of general rooted maps with a cotree marked with vertex partition given.

Theorem 13.10 The number of rooted general maps with a cotree marked for a vertex(root-vertex included) partition $\underline{n}$ given is

$$|\mathcal{M}(l;\underline{n})| = \frac{(n+2l-2)!}{(2l-1)!\underline{n}!}, \tag{13.26}$$

where l is the Betti number(the size of cotree) and $n = |\underline{n}|$.

Proof A direct result of Lemma 13.17 and Theorem 13.2. $\square$

XIII.6 Pan-flowers

A map is called a *pan-flower* if it can be seen as a standard petal bundle added a tree such that only all vertices are in the inner parts of edges on the petal bundle. The petal bundle is seen as the boundary of a pan-flower. Because of not a circuit for the base graph of a petal bundle in general, a pan-flower is reasonably seen as a generalization of a map with a boundary, or a boundary map. A pan-Halin map is only a special case when the petal bundle is asymmetric.

For convenience, the petal bundle in a pan-flower is called the *base map*. If a edge of a fundamental circuit on the underlying graph of base graph is allowed to have no articulate vertex of the tree, then the pan-flower is said to be *pre-standard*. If the edge has at least one articulate vertex of the tree, then the pan-flower is *standard*.

This section is concerned with pan-flowers in the two classes foe a vertex partition vector given.

Let $\mathcal{H}_{\mathrm{psH}}$ be the set of all rooted pre-standard pan-flowers, where the root is chosen to be an element incident to the vertex of base map. For any $H = (\mathcal{X}, \mathcal{J}) \in \mathcal{H}_{\mathrm{psH}}$, the tree T_H is always seen as a planted tree whose root is first encountered on the rooted face of H starting from the root of H. Otherwise, the first encountered at the root- vertex of H from the root.

Lemma 13.18 Let $\mathcal{H}_{\mathrm{psH}}(p; \underline{s})$ be the set of all rooted pre-standard pan-flowers with vertex partition vector $\underline{s} = (s_2, s_3, \cdots)$ on a surfaces of orientable genus p, then

$$|\mathcal{H}_{\mathrm{psH}}(p; \underline{s})| = 2^{j_1 - \delta_{p,1} + 2} \binom{j_1 + 2p}{2p - 1} |\mathcal{T}_1(\underline{j})|, \tag{13.27}$$

where $\mathcal{T}_1(\underline{j})$ is the set of planted trees with vertex partition vector $\underline{j} = (j_1, j_2, \cdots)$, such that $s_i = j_i$, $i \neq 3$; $s_3 = j_1 + j_3 + 1$, $p \geq 1$, $\underline{s} \geq \underline{0}$, $\underline{j} \geq \underline{0}$, but $\underline{s} \neq \underline{0}$ and $\underline{j} \neq \underline{0}$. Further, $\delta_{p,1}$ is the Kronecker symbol, *i.e.*, $\delta_{p,1} = 1$, when $p = 1$; $\delta_{p,1} = 0$, otherwise.

Proof On the basis of pre-standardness, it is from the definition of pan-flowers seen that an element in the set on the left hand

side of (13.27) has an element in the set on the right hand side in correspondence.

In what follows, to prove that each $T = (\mathcal{X}, \mathcal{J}) \in \mathcal{T}_1(\underline{j})$, produces $2^{j_1-\delta_{p,1}+2}\binom{j_1+2p}{2p-1}$ maps in $\mathcal{H}_{\mathrm{psH}}(p;\underline{s})$.

Denoted by (r), (x_1), (x_2), $\cdots$, (x_{j_1}) all the articulate vertices of T, where $0 < l_1 < l_2 < \cdots < l_{j_1}$, such that $x_i = (\mathcal{J}\alpha\beta)^{l_i} r$, $i = 1, , 2, \cdots, j_1$, $r = r(T)$ is the root of T.

First, by considering that the underlying graph of base map has $2p$ loops, only one vertex, its embedding on the orientable surface of genus $2p$ has exactly one face. Because of the order of its automorphism group 8 when $p = 1$, only one possible way; $2p$ when $p \geq 2$, two possible ways.

Then, the assignment of the $j_1 + 1$ articulate vertices, (r), (x_i), $i = 1, 2, \cdots, j_1$, of T on the base map has the number of ways as choosing $2p - 1$ from $j_1 + 2$ intervals with repetition allowable. That is

$$\binom{j_1+2+(2p-1)-1}{2p-1} = \binom{j_1+2p}{2p-1}.$$

Finally, since each of elements r, x_i, $i = 1, 2, \cdots, j_1$, has 2 ways: one side $\{\alpha x, \alpha\beta x\}$, or the other $\{x, \beta x\}$, they have

$$2^{j_1+1}$$

ways altogether.

In summary of the three cases, The aim reaches at. $\square$

On the basis of this lemma, by employing a result in §13.1, the following theorem can be deduced.

Theorem 13.11 The number of pre-standard rooted pan-flowers with vertex partition vector $\underline{s} = (s_2, s_3, \cdots)$ and its base involving m missing vertices on an orientable surface of genus p, $p \geq 1$, is

$$2^{m-\delta_{p,1}+1}\binom{m+2p-1}{2p-1}\binom{s_3}{m}\frac{n!m}{\underline{s}!}, \qquad (13.28)$$

where $\underline{s}! = \prod_{i\geq 2} s_i! = s_2!s_3!\cdots$ and $n + 2 = \sum_{i\geq 2} s_i$.

Proof From Lemma 13.18, this number is

$$2^{m-\delta_{p,1}+1}\binom{m+2p-1}{2p-1}\tau_1(\underline{j}),$$

where $\underline{j}=(j_1,j_2,j_3,\cdots)$ such that $j_1+1=m$, $j_3=s_3-m$ and $j_i=s_i$, $i\neq 3$, $i\geq 2$. Then from Theorem 13.2, we have

$$\begin{aligned}\tau(\underline{j})&=\frac{(n'-1)!}{\underline{j}!}=\frac{n!}{(m-1)!s_2!(s_3-m)!s_4!\cdots}\\&=\binom{s_3}{m}\frac{n!m}{\underline{s}!},\end{aligned}$$

where $n=n'-1=\sum_{i\geq 1}j_i-1=\sum_{i\geq 2}s_i-2$. By substituting this into the last, (13.28) is obtained. □

Let $\tilde{\mathcal{H}}_{\mathrm{psH}}(q;\underline{s})$ be the set of all pre-standard rooted pan-flowers with vertex partition vector $\underline{s}=(s_2,s_3,\cdots)$ on a nonorientable surface of genus q, $\underline{s}\geq\underline{0}$, but $\underline{s}\neq\underline{0}$.

Lemma 13.19 For $\tilde{\mathcal{H}}_{\mathrm{psH}}(q;\underline{s})$, $q>0$, $\underline{s}\geq\underline{0}$, but $\underline{s}\neq\underline{0}$, we have

$$|\tilde{\mathcal{H}}_{\mathrm{psH}}(q;\underline{s})|=2^{m-\delta_{q,1}+1}\binom{m+q-1}{q-1}|\mathcal{T}_1(\underline{j})|, \tag{13.29}$$

where $\underline{j}=(m-1,s_2,s_3-m,s_4,s_5,\cdots)$ and m is the number of trivalent vertices($i.e.$, missing vertices on its base).

Proof 与 Similarly to the proof of Lemma 13.18. However, an attention should be paid to that the size of the base is q, $q\geq 1$, instead of $2p$, $p\geq 1$ and the order of automorphism group of the base is $2q$ when $q\geq 2$; 4 when $q=1$. □

Similarly to Theorem 13.11, we have

Theorem 13.12 The number of pre-standard rooted pan-flowers with vertex partition vector $\underline{s}=(s_2,s_3,\cdots)$ and its base involving m missing vertices on a nonorientable surface of genus q, $q\geq 1$, is

$$2^{m-\delta_{q,1}+1}\binom{m+q-1}{q-1}\binom{s_3}{m}\frac{n!m}{\underline{s}!}, \tag{13.30}$$

where $n+2=\sum_{i\geq 2} s_i$ and $\underline{s}!=\prod_{i\geq 2} s_i!$.

Proof Similarly to the proof of Theorem 13.11, from Lemma 13.19 and Theorem 13.2, the theorem is done. $\square$

For standard pan-flowers, let $\mathcal{H}_{\mathrm{sH}}$ be the set of all such maps. From the definition, if the base map has m missing vertices, then it is not possible on an orientable surface of genus greater then $m/2$, or on a nonorientable surface of genus greater than m.

Lemma 13.20 Let $\mathcal{H}_{\mathrm{sH}}(p;\underline{s})$ be the set of all standard rooted pan-flowers with vertex partition vector $\underline{s}=(s_2,s_3,\cdots)$ on an orientable surface of genus p. If maps in $\mathcal{H}_{\mathrm{sH}}(p;\underline{s})$ have its base with $m\geq 2p$ missing vertices, then we have

$$|\mathcal{H}_{\mathrm{sH}}(p;\underline{s})|=2^{m-\delta_{p,1}+1}\binom{m-1}{2p-1}|\mathcal{T}_1(\underline{j})|, \tag{13.31}$$

where $\mathcal{T}_1(\underline{j})$, as above, is the set of panted trees with vertex partition vector $\underline{j}=(j_1,j_2,j_3,\cdots)$ for $j_1=m-1$, $j_3=s_3-m$, $j_i=s_i$, $i\neq 3$, $i\geq 2$.

Proof For any $H\in\mathcal{H}_{\mathrm{sH}}(p;\underline{s})$, from the definitions of standardness and pan-flowers, it is seen that there exists a planted tree in $\mathcal{T}_1(\underline{j})$ corresponding to H. Thus, it suffices to prove that any planted tree in $T=(\mathcal{X},\mathcal{J})\in\mathcal{T}_1(\underline{j})$ produces

$$2^{m-\delta_{p,1}+1}\binom{m-1}{2p-1}$$

maps in $\mathcal{H}_{\mathrm{sH}}(p;\underline{s})$.

First, an attention should be paid to that maps in $\mathcal{H}_{\mathrm{sH}}(p;\underline{s})$ are with their base of size $2p$ on an orientable surface of genus p. Since the order of automorphism group of the base is $4p$ when $p\geq 2$; 8 when $p=1$, the base has, respectively, 2 ways when $p\geq 2$; 1 way to choose its root.

Second, since the number of missing vertices on the base is m, T must have $m-1$ unrooted articulate vertices. Let them be incident with (x_1), $\cdots$, (x_{m-1}). From the standardness again, there are $m-$

1 intervals for choice in the linear order $\langle (r), (x_1), (x_2), \cdots, (x_{m-1})\rangle$. Thus, any $2p-1$ points insertion divides the linear order into $2p$ nonempty segments. This has

$$\binom{m-1}{2p-1}$$

distinct ways.

Third, notice that each of the m articulate edges including the root-edge in T has 2 choices, and hence

$$2^m$$

distinct choices altogether.

In summary rom the three cases, the lemma is soon found. □

Based on this, we have

Theorem 13.13 The number of standard rooted pan-flowers with vertex partition vector $\underline{s} = (s_2, s_3, \cdots)$ and their base map of m, $m \geq 2p$, unrooted vertices on an orientable surface S_p of genus $p \geq 1$, is

$$2^{m-\delta_{q,1}+1}\binom{m-1}{2p-1}\binom{s_3}{m}\frac{n!m}{\underline{s}!}, \tag{13.32}$$

where $n+2 = \sum_{i\geq 2} s_i$.

Proof Similarly to the proof of Theorem 13.11. However, by Lemma 13.20 instead of Lemma 13.8. □

At a look again for the nonorientable case.

Lemma 13.21 Let $\tilde{\mathcal{H}}_{\mathrm{sH}}(q;\underline{s})$ be the set of standard rooted pan-flowers with vertex partition vector $\underline{s} = (s_2, s_3, \cdots)$ on a nonorientable surface of genus $q \geq 1$. If each map in $\tilde{\mathcal{H}}_{\mathrm{sH}}(q;\underline{s})$ has its base map of m unrooted vertices, then we have

$$|\tilde{\mathcal{H}}_{\mathrm{sH}}(q;\underline{s})| = 2^{m-\delta_{q,1}+1}\binom{m-1}{q-1}\tau(\underline{j}), \tag{13.33}$$

where $\tau_1(\underline{j}) = |\mathcal{T}_1(\underline{j})|$, $\underline{j} = (j_1, j_2, j_3, \cdots)$, $j_1 = m-1$, $j_3 = s_3 - m$, $j_i = s_i$, $i \neq 3$, $i \geq 2$.

Proof Similarly to the proof of Lemma XII.2.3. However, what an attention should be paid to is the base of size q instead of $2p$ for a surface of genus q. □

Thus, we can also have

Theorem 13.14 The number of standard rooted pan-flowers with vertex partition vector $\underline{s} = (s_2, s_3, \cdots)$ and their bases of m unrooted vertices on a nonorientable surface of genus q, $q \geq 1$, is

$$2^{m-\delta_{q,1}+1}\binom{m-1}{q-1}\binom{s_3}{m}\frac{n!m}{\underline{s}!}, \tag{13.34}$$

where $\underline{s} \geq \underline{0}$, $\underline{s} \neq \underline{0}$, $m \geq q \geq 1$ and $n+2 = \sum_{i\geq 2} s_i$.

Proof Similarly to the proof of Theorem 13.13. However, by Lemma 13.21 instead of by Lemma 13.20. □

Activities on Chapter XIII

XIII.7 Observations

O13.1 Observe the number of plane rooted trees of size $n \geq 0$ given.

O13.2 Observe the number of outerplanar rooted maps of size $n \geq 0$ given.

O13.3 Observe the number of wintersweets of size $n \geq 0$.

O13.4 To show a relationship between outer planar maps and trees.

O13.5 Observe how to evaluate the number of plane rooted trees with the number of articulate vertices $m \geq 2$ and size $n \geq 1$ given.

O13.6 Consider what relation have the enufunction with the number of nonrooted vertices, nonrooted faces and the enufunction of vertex partition and genus as parameters.

O13.7 Observe the number of rooted plane tree with root-vertex valency and vertex partition.

O13.8 Observe the number of rooted plane trees with the number of articulate vertices and vertex partition of other vertices.

O13.9 Observe the difference between boundary maps and non-boundary maps.

O13.10 observe the automorphism groups of a map and one of its boundary map by .

XIII.8 Exercises

E13.1 Determine the enufunction of planted trees with vertex partition as parameters by establishing and solving a equation.

A *wintersweet* is a rooted map with the property that it becomes a tree if missing circuits only at nonrooted terminal vertices of the tree.

E13.2 Establish an equation satisfied by the vertex partition function of Wintersweets and then try to solve the equation.

A rooted map is called *unicyclic* if it has only one circuit.

E13.3 Establish an equation satisfied by the vertex partition function of unicyclic maps when the root is on a circuit and then try to solve the equation.

E13.4 Establish an equation satisfied by the vertex partition function of Halin rooted maps and then try to solve the equation.

E13.5 Establish an equation satisfied by the vertex partition function of outerplanar rooted maps and then try to solve the equation.

E13.6 Establish an equation satisfied by the face partition function of outerplanar rooted maps when the root is not on the circuit and then try to solve the equation.

E13.7 Establish an equation satisfied by the face partition function of planar rooted petal bundles and then try to solve the equation.

E13.8 Establish an equation satisfied by the face partition function of outerplanar rooted maps when the root is on the circuit and then try to solve the equation.

E13.9 Establish an equation satisfied by the vertex partition function of unicyclic maps when the root is not on a circuit and then try to solve the equation.

E13.10 Establish an equation satisfied by the face partition function of planar rooted supermaps of bouquets and then try to solve

the equation.

E13.11 Establish an equation satisfied by the vertex partition function of planar rooted maps with two vertex disjoint circuits and then try to solve the equation.

XIII.9 Researches

R13.1 Determine the vertex partition function of general rooted maps on the sphere.

R13.2 Determine the vertex partition function of general rooted maps on all surfaces.

R13.3 Determine the vertex partition function of general Eulerian rooted maps on all surfaces.

R13.4 Determine the vertex partition function of 2-edge connected rooted maps on the sphere.

R13.5 Determine the vertex partition function of 2-edge connected rooted maps on all surfaces.

R13.6 Determine the vertex partition function of nonseparable rooted maps on the sphere.

R13.7 Determine the vertex partition function of nonseparable rooted maps on all surfaces.

R13.8 Determine the vertex partition function of loopless rooted maps on the sphere.

R13.9 Determine the vertex partition function of loopless rooted maps on all surfaces.

R13.10 Determine the vertex partition function of rooted triangulations on all surfaces.

Chapter XIV

Super Maps of a Graph

- A semi-automorphism of a graph is a bijection from its semiedge set to itself generated by the binary group sticking on all edges such that the partitions in correspondence.

- An automorphism of a graph is a bijection from the edge set to itself such that the adjacency on edges in correspondence.

- The semi-automorphism group of a graph is different from its automorphism group if, and only if, a loop occurs.

- Nonisomorphic super rooted and unrooted maps of a graph can be done from the embeddings of the graph via its automorphism group or semi-automorphism group of the graph.

XIV.1 Semi-automorphisms on a graph

A pregraph is considered as a partition on the set of all semiedges as shown in Chapter I. Let $G = (\mathcal{X}, \delta; \pi)$ be a pregraph where $\mathcal{X}$, δ and π are, respectively, the set of all semiedges, the permutation determined by edges and the partition on $\mathcal{X}$.

Two regraphs $G_1 = (\mathcal{X}_1, \delta_1; \pi_1)$ and $G_2 = (\mathcal{X}_2, \delta_2; \pi_2)$ are said to

be *seme-isomorphic* if there is a bijection $\tau : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ such that

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\quad\tau\quad} & \mathcal{X}_2 \\ {\scriptstyle\gamma_1}\big\downarrow & & \big\downarrow{\scriptstyle\gamma_2} \\ \gamma_1(\mathcal{X}_1) & \xrightarrow{\quad\tau_{\gamma_1}\quad} & \gamma_2(\mathcal{X}_2) \end{array} \tag{14.1}$$

are commutative for $\gamma = \delta$ and π where τ_{γ_1} is induced from τ on $\gamma_1(\mathcal{X}_1)$. The bijection τ is called a *semi-automorphism* between G_1 and G_2.

Example 14.1 Given two pregraphs $G = (\mathcal{X}, \delta_1; \pi_1)$ where

$$\begin{cases} \mathcal{X} = \sum_{i=1}^{8} \{x_i(0), x_i(1)\},\ \delta_1 = \prod_{i=1}^{8} (x_i(0), x_i(1)); \\ \pi_1 = \{X_i |\ 1 \le i \le 8\} \end{cases}$$

with

$$\begin{cases} X_1 = \{x_1(0), x_6(0), x_6(1)\},\ X_2 = \{x_1(1), x_2(0), x_3(0)\}, \\ X_3 = \{x_2(1), x_5(0), x_7(0)\},\ X_4 = \{x_4(0), x_5(1), x_7(1), x_8(0)\}, \\ X_5 = \{x_4(1), x_5(1), x_8(1)\} \end{cases}$$

and $H = (\mathcal{Y}, \delta_2; \pi_2)$ where

$$\begin{cases} \mathcal{Y} = \sum_{i=1}^{8} \{y_i(0), y_i(1)\},\ \delta_2 = \prod_{i=1}^{8} (y_i(0), y_i(1)); \\ \pi_2 = \{Y_i |\ 1 \le i \le 8\} \end{cases}$$

with

$$\begin{cases} Y_1 = \{y_4(0), y_6(0), y_7(0)\},\ Y_2 = \{y_5(0), y_6(1), y_7(1), y_8(0)\}, \\ Y_3 = \{y_3(0), y_5(1), y_8(1)\},\ Y_4 = \{y_2(0), y_3(1), y_4(1)\}, \\ Y_5 = \{y_1(0), y_1(1), y_2(1)\}. \end{cases}$$

Let $\tau :\ \mathcal{X} \longrightarrow \mathcal{Y}$ be a bijection with (14.1) commutative for $\gamma_i = \delta_i$, $i = 1$ and 2, as

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ \delta_2 y_2 & \delta_2 y_4 & y_3 & y_8 & y_6 & y_1 & y_7 & y_5 \end{pmatrix}.$$

Because of

$$\begin{cases} \tau X_1 = \{\tau x_1(0), \tau x_6(0), \tau x_6(1)\} = \{y_2(1), y_1(0), y_1(1)\} = Y_5, \\ \tau X_2 = \{\tau x_1(1), \tau x_2(0), \tau x_3(0)\} = \{y_2(0), y_4(1), y_3(1)\} = Y_4, \\ \tau X_3 = \{\tau x_2(1), \tau x_5(0), \tau x_7(0)\} = \{y_4(0), y_6(0), y_7(0)\} = Y_1, \\ \tau X_4 = \tau\{x_4(0), x_5(1), x_7(1), x_8(0)\} \\ \qquad\qquad\qquad = \{y_6(1), y_7(1), y_5(0), y_8(0)\} = Y_2, \\ \tau X_5 = \{\tau x_3(1), \tau x_4(1), \tau x_8(1)\} = \{y_3(0), y_8(1), y_5(1)\} = Y_3, \end{cases}$$

we have $\tau_{\pi_1}\pi_1 = \pi_2\tau$, *i.e.*, (14.1) is commutative for $\gamma_i = \pi_i$, $i = 1$ and 2. Therefore, τ is a semi-isomorphism between G and H.

Lemma 14.1 If two pregraphs G and H are semi-isomorphic, then they have the same number of connected components provided omission of isolated vertex.

Proof By contradiction. Suppose G and H are semi-isomorphic with a semi-isomorphism $\tau : G \to H$ but $G = G_1 + G_2$ with two components: G_1 and G_2 and H, a component itself. From the the commutativity of (14.1), H has two components as well. This contradicts to the assumption that H is a component itself. □

If $G = H$, then a semi-isomorphism between G and H is called a *semi-automorphism* of G. Lemma 14.1 enables us to discuss semi-automorphism of only a graph instead of a pregraph without loss generality.

Lemma 14.1 allows us to consider only graphs instead of pregraphs for semi-automorphisms.

Moreover for the sake of brevity, only graphs of order greater than 4 are considered as the general case in what follows.

Theorem 14.1 The set of all semi-automorphisms of a graph forms a group.

Proof Because of all semi-automorphisms as permutations acting on the set of semi-edges, the commutativity leads to the closedness in the set of all semi-automorphisms under composition with the as-

sociate law. Moreover, easy to check that the identity permutation is a semi-automorphisms and the inverse of a semi-automorphism is still a semi-automorphism. This theorem holds. □

This group in Theorem 14.1 is called the *semi-automorphism group* of the graph.

Example 14.2 In Example 14.1, the pregraph $G = (\mathcal{X}, \delta_1; \pi_1)$ is a graph. It is easily checked that

$$\tau_1 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ x_1 & x_2 & x_3 & x_4 & x_5 & \delta_1 x_6 & x_7 & x_8 \end{pmatrix} = (x_6(0), x_6(1))$$

is a semi-automorphism on G. It can also be checked that the semi-automorphism group of G is $\mathrm{Aut}_{\mathrm{hf}}(G) = \{\tau_i | 0 \leq i \leq 11\}$ where

$$\begin{cases} \tau_0 = 1, \text{ the identity;} \\ \tau_1 = (x_5, x_7); \\ \tau_2 = (x_4, x_8); \\ \tau_3 = (x_5, x_7)(x_4, x_8); \\ \tau_4 = (x_2, x_3)(x_4, \delta_1 x_5)(x_7, \delta_1 x_8); \\ \tau_5 = (x_2, x_3)(x_5, \delta_1 x_8)(x_7, \delta_1 x_4); \\ \tau_i = (x_6(0), x_6(1))\tau_{i-6}, \ 6 \leq i \leq 11. \end{cases}$$

XIV.2 Automorphisms on a graph

Now, let us be back to the usual form of a graph $G = (V, E)$ where V and E are, respectively, the vertex and edge sets. In fact, if X_i as described in §XIV.1 is denoted by v_i, then $V = \{v_0 | i = 0.1.2. \cdots\}$ and $E = \{x_j | j = 0, 1, 2, \cdots\}$.

An *edge-isomorphism* of two pregraphs $G_i = (V_i, E_i)$, $i = 1$ and

2, is defined as a bijection $\tau: \ E_1 \longrightarrow E_2$ with diagram

$$\begin{tikzcd} E_1 \arrow[r, "\tau"] \arrow[d, "\eta_1"'] & E_2 \arrow[d, "\eta_2"] \\ V_1 \arrow[r, "\tau_{\eta_1}"] & V_2 \end{tikzcd} \tag{14.2}$$

commutative where η_i, $i = 1$ and 2, are seen a mapping $2^{E_i} \longrightarrow V_i$.

When $G = G_1 = G_2$, an edge-isomorphism between G_1 and G_2 becomes an *edge-automorphism* on G.

Lemma 14.2 If two pregraphs G and H are edge-isomorphic, then they have the same number of components provided omission of isolated vertex.

Proof Similar to the proof of Lemma 14.1. □

This lemma enables us to discuss only graphs instead of pregraphs for edge-isomorphisms or edge-automorphisms.

Theorem 14.2 All edge-automorphisms of a graph G forms a group, denoted by $\mathrm{Aut_{ee}(G)}$.

Proof Similar to the proof of Theorem 14.1. □

Example 14.3 The graph G In Example 14.2 has its $\mathrm{Aut_{ee}}(G) = \{\tau_i | 0 \le i \le 5\}$ where

$$\begin{cases} \tau_0 = 1, \ \text{the identity}; \\ \tau_1 = (x_5, x_7); \\ \tau_2 = (x_4, x_8); \\ \tau_3 = (x_5, x_7)(x_4, x_8); \\ \tau_4 = (x_2, x_3)(x_4, x_5)(x_7, x_8); \\ \tau_5 = (x_2, x_3)(x_5, x_8)(x_7, x_4). \end{cases}$$

An *isomorphism*, or in the sense above *vertex-isomorphism*, between two pregraphs $G_i = (V_i, E_i)$, $i = 1$ and 2, is defined as a bijection

$\tau: \ V_1 \longrightarrow V_2$ which satisfies that the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\tau} & V_2 \\ \xi_1 \downarrow & & \downarrow \xi_2 \\ E_1 \subseteq V_1 \times V_1 & \xrightarrow{\tau_{\xi_1}} & E_2 \subseteq V_2 \times V_2 \end{array} \tag{14.3}$$

is commutative where $\xi_i(v_i) = E_{iv_i}$ for $v_i \in V_i$, $i = 1$ and 2.

When $G = G_1 = G_2$, a isomorphism between G_1 and G_2 is called an *automorphism* of G.

Lemma 14.3 If two pregraphs G and H are isomorphic, then they have the same number of components.

Proof Similar to the proof of Lemma 14.2. $\square$

This lemma enables us to discuss only graphs instead of pregraphs for isomorphisms or automorphisms.

Theorem 14.3 all automorphisms of a graph G form a group, denoted by Aut(G).

Proof Similar to the proof of Theorem 14.2. $\square$

The group mentioned in this theorem is called the *automorphism group* of G.

Example 14.4 The graph G In Example 14.2 has its $\mathrm{Aut}(G) = \{\tau_i | 0 \leq i \leq 1\}$ where

$$\begin{cases} \tau_0 = 1, \text{ the identity;} \\ \tau_1 = (X_3, X_5). \end{cases}$$

Because of no influence on the automorphism group of a graph when deleting loops, or replacing multiedge by a single edge, τ_i, $i = 1, 2$ and 3 in Example 14.3 are to the identity τ_0 and τ_4 and τ_5 in Example 14.3 to τ_1 here.

XIV.3 Relationships

Fundamental relationships among those groups mentioned in the last section are then explained for the coming usages.

Theorem 14.4 $\mathrm{Aut_{hf}(G)} \sim \mathrm{Aut_{ee}(G)}$ if, and only if, G is loopless.

Proof Necessity. By contradiction. Suppose $\mathrm{Aut_{hf}(G)} \sim \mathrm{Aut_{ee}(G)}$ with an edge-automorphism τ but G has a loop denoted by $z = (z(0), z(1))$. Assume $\tau(l) = l$ without loss of generality. However, both the semi-automorphisms: τ_1 and τ_2 corresponding to τ are found as

$$\tau_1(x) = \begin{cases} \tau(x), & \text{when } x \neq z; \\ z(0), & \text{when } x = z(0); \\ z(1), & \text{when } x = z(1) \end{cases}$$

and

$$\tau_2(x) = \begin{cases} \tau(x), & \text{when } x \neq z; \\ z(0), & \text{when } x = z(1); \\ z(1), & \text{when } x = z(0). \end{cases}$$

This implies $\mathrm{Aut_{hf}(G)} \not\sim \mathrm{Aut_{ee}(G)}$, a contradiction.

Sufficiency. Because of no loop in G, the symmetry between two ends of a link leads to $\mathrm{Aut_{hf}(G)} \sim \mathrm{Aut_{ee}(G)}$. □

From the proof of Theorem 14.4, the following corollary can be done.

Corollary 14.1 Let l be the number of loops in G, then

$$\mathrm{Aut_{hf}}(G) \sim S_2^l \times \mathrm{Aut_{ee}}(G)$$

where S_2 is the symmetric group of order 2.

Proof Because of exact two semi-automorphisms deduced from an edge-automorphism and a loop, the conclusion is done. □

From this corollary, we can soon find

$$\mathrm{aut_{hf}}(G) = 2^l \times \mathrm{aut_{ee}}(G). \tag{14.4}$$

Because of no contribution of a loop to the automorphism group of G, the graph G has its automorphism group Aut(G) always for that obtained by deleting all loops on G.

Theorem 14.5 $\mathrm{Aut_{ee}(G)} \sim \mathrm{Aut(G)}$ if, and only if, G is simple.

Proof Because of no contribution of either loops or multiedges to the automorphism group Aut(G), the theorem holds. □

In virtue of the proof of Theorem 12.5, a graph with multi-edges G has its automorphism group Aut(G) always for its underlying simple graph, *i.e.*, one obtained by substituting a link for each multi-edge on G.

Lemma 14.4 Let G be a graph with i-edges of number m_i, $i \geq 2$. Then, its edge-automorphism group

$$\mathrm{Aut_{ee}(G)} = \sum_{\mathrm{i}\geq 2} \mathrm{m_i S_i} \times \mathrm{Aut(G)}$$

where S_i is the symmetric group of order i, $i \geq 2$.

Proof In virtue of S_m as the edge-automorphism group of link bundle P_m of size m, $m \geq 2$, the lemma is done. □

On the basis of Lemma 14.4, we can obtain

Corollary 14.2 Let l and m_i be, respectively, the number of loops and i-edges, $l \geq 1, i \geq 2$ in G, then

$$\mathrm{aut_{hf}}(G) = 2^l n_{\mathrm{me}} \mathrm{aut}(G) \tag{14.5}$$

where

$$n_{\mathrm{me}} \equiv n_{\mathrm{me}}(G) = \sum_{i\geq 1} i! m_i$$

which is called the *multiplicity* of G.

Proof By considering Corollary 14.1, the conclusion is done. □

XIV.4 Nonisomorphic super maps

For map $M = (\mathcal{X}_{\alpha,\delta}, \mathcal{P})$, its automorphisms are discussed with asymmetrization in Chapter VIII. Let $\mathcal{M}(G)$ be the set of all nonisomorphic maps with underlying graph G.

Lemma 14.5 For an automorphism ζ on map $M = (\mathcal{X}_{\alpha,\delta}(X), \mathcal{P})$, we have exhaustively

$$\zeta|_{\delta(X)} \in \mathrm{Aut_{hf}(G)} \text{ and } \zeta\alpha|_{\delta(\mathrm{X})} \in \mathrm{Aut_{hf}(G)}$$

where $G = G(M)$, the under graph of M, and $\delta(X) = X + \delta X$.

Proof Because $\mathcal{X}_{\alpha,\delta}(X) = (X + \delta X) + (\alpha X + \alpha\delta X) = \delta(X) + \alpha\delta(X)$, by Conjugate Axiom each $\zeta \in \mathrm{Aut(M)}$ has exhaustively two possibilities: $\zeta|_{\delta(X)} \in \mathrm{Aut_{hf}(G)}$ and $\zeta\alpha|_{\delta(X)} \in \mathrm{Aut_{hf}(G)}$. □

On the basis of Lemma 14.5, we can find

Theorem 14.6 Let $\mathcal{E}_g(G)$ be the set of all embeddings of a graph G on a surface of genus g(orientable or nonorientable), then the number of nonisomorphic maps in $\mathcal{E}_g(G)$ is

$$m_g(G) = \frac{1}{2 \times \mathrm{aut_{hf}(G)}} \sum_{\tau \in \mathrm{Aut_{hf}(G)}} |\Phi(\tau)| \tag{14.6}$$

where $\Phi(\tau) = \{M \in \mathcal{E}_g(G) | \tau(M) = M \text{ or } \tau\alpha(M) = M\}$.

Proof Suppose X_1, X_2, $\cdots$, X_m are all the equivalent classes of $X = \mathcal{E}_g(G)$ under the group $\mathrm{Aut_{hf}(G)} \times \langle\alpha\rangle$, then $m = m_g(G)$. Let

$$S(x) = \{\tau \in \mathrm{Aut_{hf}(G)} \times \langle\alpha\rangle |\ \tau(\mathrm{x}) = \mathrm{x}\}$$

be the stabilizer at x, a subgroup of $\mathrm{Aut_{hf}(G)} \times \langle\alpha\rangle$. Because

$$|\mathrm{Aut_{hf}(G)} \times \langle\alpha\rangle| = |\mathrm{S(x_i)}||\mathrm{X_i}|,$$

$x_i \in X_i$, $i = 1, 2, \cdots, m$, we have

$$m|\mathrm{Aut_{hf}(G)} \times \langle\alpha\rangle| = \sum_{\mathrm{i=1}}^{\mathrm{m}} |\mathrm{S(x_i)}||\mathrm{X_i}|. \tag{1}$$

By observing $|S(x_i)|$ independent of the choice of x_i in the class X_i, the right hand side of (1) is

$$\begin{aligned}\sum_{x\in X}|S(x)| &= \sum_{x\in X}\sum_{\tau\in S(x)}1\\ &= \sum_{\tau\in \mathrm{Aut}_{\mathrm{hf}}(\mathrm{G})\times\langle\alpha\rangle}\ \sum_{x=\tau(x)}1 \qquad (2)\\ &= \sum_{\tau\in \mathrm{Aut}_{\mathrm{hf}}(\mathrm{G})\times\langle\alpha\rangle}|\Phi(\tau)|.\end{aligned}$$

From (1) and (2), the theorem can be soon derived. □

The theorem above shows how to find nonisomorphic super maps of a graph when the semi-automorphism group of the graph is known.

Theorem 14.7 Let G be a graph with l loops and m_i multi-edges of multiplier i and $\mathcal{E}_g(G)$, the set of all embeddings of G on a surface of genus g(orientable or nonorientable), then the number of nonisomorphic maps in $\mathcal{E}_g(G)$ is

$$m_g(G) = \frac{1}{2^{l+1}n_{\mathrm{me}}\mathrm{aut}(\mathrm{G})}\sum_{\tau\in \mathrm{Aut}_{\mathrm{hf}}(\mathrm{G})}|\Phi(\tau)| \qquad (14.7)$$

where $\Phi(\tau) = \{M \in \mathcal{E}_g(G)|\tau(M) = M \text{ or } \tau\alpha(M) = M\}$ and n_{me} is the multiplicity of G.

Proof On the basis of Theorem 14.6, the conclusion is soon derived from Corollary 14.2. □

Corollary 14.3 Let G be a simple graph. Then, the number of nonisomorphic maps in $\mathcal{E}_g(G)$ is

$$m_g(G) = \frac{1}{2\mathrm{aut}(\mathrm{G})}\sum_{\tau\in \mathrm{Aut}_{\mathrm{hf}}(\mathrm{G})}|\Phi(\tau)| \qquad (14.8)$$

where $\Phi(\tau) = \{M \in \mathcal{E}_g(G)|\tau(M) = M \text{ or } \tau\alpha(M) = M\}$ and n_{me} is the multiplicity of G.

Proof A direct result of Theorem 14.7 via considering G with neither loop nor multi-edge. □

XIV.5 Via rooted super maps

Another approach for determining nonisomorphic super maps of a graph is via rooted ones whenever its distinct embeddings are known.

Theorem 14.8 For a graph G, let $\mathcal{R}_g(G)$ and $\mathcal{E}_g(G)$ be, respectively, the sets of all nonisomorphic rooted super maps and all distinct embeddings of G with size $\epsilon(G)$ on a surface of genus g(orientable or nonorientable). Then,

$$|\mathcal{R}_g(G)| = \frac{2\epsilon(G)}{\mathrm{aut_{hf}(G)}}|\mathcal{E}_g(G)|. \tag{14.9}$$

Proof Let $\mathcal{M}_g(G)$ be the set of all nonisomorphic super maps of G. By (11.3), we have

$$|\mathcal{R}_g(G)| = \sum_{M\in\mathcal{M}_g(G)} \frac{4\epsilon(G)}{\mathrm{aut(M)}}$$

i.e.,

$$\frac{4\epsilon(G)}{2\times \mathrm{aut_{hf}(G)}} \sum_{M\in\mathcal{M}_g(G)} \frac{2\times \mathrm{aut_{hf}(G)}}{\mathrm{aut(M)}}.$$

By considering that $2\times \mathrm{aut_{hf}(G)} = |\mathrm{Aut_{hf}(G)}\times\langle\alpha\rangle|$ is

$$|(\mathrm{Aut_{hf}(G)}\times\langle\alpha\rangle)|_{\mathrm{M}}| \times |\mathrm{Aut_{hf}(G)}\times\langle\alpha\rangle(\mathrm{M})|$$

and $(\mathrm{Aut_{hf}(G)}\times\langle\alpha\rangle)|_{\mathrm{M}} = \mathrm{Aut(M)}$ from Lemma 14.5, we have $|\mathcal{R}_g(G)|$ is

$$\begin{aligned}&\frac{4\epsilon(G)}{|\mathrm{Aut_{hf}(G)}\times\langle\alpha\rangle|} \sum_{M\in\mathcal{M}_g(G)} |\mathrm{Aut_{hf}(G)}\times\langle\alpha\rangle(M)|\\ &= \frac{2\epsilon(G)}{\mathrm{aut_{hf}(G)}}|\mathcal{E}_g(G)|.\end{aligned}$$

This is (14.9). □

This theorem enables us to determine all the super rooted maps of a graph when the semi-automorphism group of the graph is known.

Theorem 14.9 For a graph G with l, $l \geq 1$, loops and m_i multi-edges of multiplier i, $i \geq 2$, let $\mathcal{R}_g(G)$ and $\mathcal{E}_g(G)$ be, respectively, the sets of all nonisomorphic rooted super maps and all distinct embeddings of G with size $\epsilon(G)$ on a surface of genus g(orientable or nonorientable). Then,

$$|\mathcal{R}_g(G)| = \frac{\epsilon(G)}{2^{l-1} n_{\mathrm{me}} \mathrm{aut(G)}} |\mathcal{E}_g(G)| \tag{14.10}$$

where n_{me} is the multiplicity of G.

Proof A direct result of Theorem 14.8 from Lemma 12.2. □

Corollary 14.4 For a simple graph G, let $\mathcal{R}_g(G)$ and $\mathcal{E}_g(G)$ be, respectively, the sets of all nonisomorphic rooted super maps and all distinct embeddings of G with size $\epsilon(G)$ on a surface of genus g(orientable or nonorientable). Then,

$$|\mathcal{R}_g(G)| = \frac{2\epsilon(G)}{\mathrm{aut(G)}} |\mathcal{E}_g(G)|. \tag{14.11}$$

Proof The case of $l = 0$ and $m_i = 0$, $i \geq 2$, of Theorem 14.9. □

Corollary 14.5 The number of rooted super maps of a simple graph G with n_i vertices of valency i, $i \geq 1$, on orientable surfaces is

$$\frac{2\epsilon}{\mathrm{aut(G)}} \prod_{i \geq 2} ((i-1)!)^{n_i} \tag{14.12}$$

where ϵ is the size of G.

Proof Because of the number of distinct embeddings on orientable surfaces

$$\sum_{g \geq 0} \mathcal{E}_g(G) = \prod_{i \geq 2} ((i-1)!)^{n_i} \tag{14.13}$$

known, Corollary 14.4 leads to the conclusion. □

Corollary 14.6 The number of rooted super maps of bouquet B_m, $m \geq 1$, on orientable surfaces is

$$\frac{(2m)!}{2^m m!}. \tag{14.14}$$

Proof Because of the number of all distinct embeddings of B_m on orientable surfaces $(2m_1)!$ and the order of its semi-automorphism group $2^m m!$ known, the conclusion is deduced from Theorem 14.8. □

In virtue of petal bundles all super maps of bouquets, (14.14) is in coincidence with (9.9).

The number of nonisomorphic super maps of a graph can also be derived from rooted ones.

Theorem 14.10 *For a given graph G, let $\mathcal{E}_k(G)$ be the set of all its nonequivalent embeddings with automorphism group order k. Then we have the number of all nonisomorphic unrooted supper maps of G is*

$$n_{\mathrm{ur}}(G) = \frac{1}{2^{l(G)+1}\mathrm{aut(G)}}\Big(\sum_{\substack{i|4\epsilon \\ 1\leq i\leq 4\epsilon}} i|\mathcal{E}_i(G)|\Big) \tag{14.15}$$

where $\epsilon = \epsilon(G)$ is the size of G and $l(G)$ is the number of loops in G.

Proof On the basis of Theorem 14.8, we have

$$|\mathcal{R}_i(G)| = \frac{2\epsilon(G)}{\mathrm{aut}_{1/2}(\mathrm{G})}|\mathcal{E}_i(G)|$$

where $\mathcal{R}_i(G)$ is rooted super maps of G with automorphism group order i. By Theorem 14.4 and Corollary 14.1,

$$|\mathcal{R}_i(G)| = \frac{\epsilon(G)}{2^{l(G)-1}\mathrm{aut(G)}}|\mathcal{E}_i(G)|.$$

Because of $4\epsilon(G)/i$ rooted maps produced by an unrooted map

in $\mathcal{R}_i(G)$ as known in the proof of Theorem 11.1, we have

$$\frac{i}{4\epsilon(G)}|\mathcal{R}_i(G)| = \frac{i}{4\epsilon(G)}\frac{\epsilon(G)}{2^{l(G)-1}\mathrm{aut(G)}}|\mathcal{E}_i(G)|$$
$$= \frac{i}{2^{l(G)+1}\mathrm{aut(G)}}|\mathcal{E}_i(G)|.$$

Overall possible $i|4\epsilon(G)$ is the conclusion of the theorem. □

Further, this theorem can be generalized for any types a set of graphs.

Theorem 14.11 *For a set of graphs $\mathcal{G}$, the number of nonisomorphic unrooted super maps of all graphs in $\mathcal{G}$ is*

$$n_{\mathrm{ur}}(\mathcal{G}) = \sum_{G\in\mathcal{G}} \frac{1}{2^{l(G)+1}\mathrm{aut(G)}} \sum_{\substack{i|4\epsilon(G)\\ 1\le i\le 4\epsilon(G)}} i|\mathcal{E}_i(G)|. \tag{14.16}$$

Proof From Theorem 14.10 overall $G \in \mathcal{G}$, the theorem is soon done. □

For a given genus g of an orientable or nonorientable surface, let $\mathcal{E}_k(G; g)$ be the set of all nonequivalent embeddings of a graph G on the surface with automorphism group order k.

Theorem 14.12 *For a given genus g of an orientable or nonorientable surface, the number of all nonisomorphic unrooted supper maps of a G on the surface is*

$$n_{\mathrm{ur}}(G; g) = \frac{1}{2^{l(G)+1}\mathrm{aut(G)}}\Big(\sum_{\substack{i|4\epsilon\\ 1\le i\le 4\epsilon}} i|\mathcal{E}_i(G; g)|\Big) \tag{14.17}$$

where $\epsilon = \epsilon(G)$ is the size of G.

Proof By classification of maps and embeddings as well with genus, from Theorem 14.11 the theorem is done. □

Furthermore, this theorem can also generalized for any types a set of graphs.

Theorem 14.13 *For a given genus g of an orientable or nonorientable surface, the number of nonisomorphic unrooted super maps of all graphs in a set of graphs $\mathcal{G}_{\mathrm{P}}$ with a given property P is*

$$n_{\mathrm{ur}}(\mathcal{G};g) = \sum_{G\in\mathcal{G}_{\mathrm{P}}} \frac{1}{4\epsilon(G)} \sum_{\substack{i|4\epsilon(G)\\ 1\le i\le 4\epsilon(G)}} i|\mathcal{E}_i(G;g)|. \tag{14.18}$$

Proof A particular case of Theorem 14.11. □

Activities on Chapter XIV

XIV.6 Observations

Let B_n be the bouquet of n loops for $n \geq 1$.

O14.1 Find the semi-automorphism group of B_n for $n \geq 1$.

O14.2 Find the edge-automorphism group of B_n for $n \geq 1$.

O14.3 Find the automorphism group of B_n for $n \geq 1$.

Let D_m be the *dipole* which is of order two and size m for $m \geq 1$ without loop.

O14.4 Show $\mathrm{Aut}_{\mathrm{hf}}(D_m) \sim \mathrm{Aut}_{\mathrm{ee}}(D_m) \sim S_m$ where S_m is the symmetric group of order m for $m \geq 1$.

O14.5 Find a condition for $\mathrm{Aut}(D_m) \sim \mathrm{Aut}_{\mathrm{hf}}(D_m)$, $m \geq 1$.

O14.6 Determine the number of super maps of K_4.

O14.7 Determine the number of rooted super maps of K_5.

O14.8 Determine the number of super maps of $K_{3,3}$ rooted and unrooted.

O14.9 Determine the number of super maps of $K_{4,4}$ rooted and unrooted.

O14.10 Suppose $A_{l,p}$ be the number of distinct embeddings of B_l, $l \geq 1$, on the orientable surface of genus $p \geq 0$, determine the number of rooted and unrooted super maps of B_l on the orientable

surface of genus $p \geq 0$.

XIV.7 Exercises

E14.1 Show that the number of rooted super maps of K_n, the complete graph of order n, $n \geq 4$, is

$$(n-2)^{n-1}.$$

E14.2 Show that the number of rooted super maps of $K_{m,n}$, the complete bipartite graph of order $m+n$, $m, n \geq 3$, is

$$2(m-1)!^{n-1}(n-1)!^{m-1}.$$

E14.3 Let $\mathcal{T}_n$ be the set of non-isomorphic trees of order n, $n \geq 2$. Show that

$$\sum_{T \in \mathcal{T}_n} \frac{\prod_{i \geq 1}(i-1)!^{n_i}}{|\mathrm{Aut(T)}|} = \frac{(2n-1)!}{n!(n+1)!},$$

the number of rooted plane trees of order $n = \sum_{i \geq 1} n_i$, where n_i is the number of vertices of valency i, $i \geq 1$.

E14.4 Determine the distribution of rooted super maps of the complete graph K_n, $n \geq 4$, by automorphism group orders.

E14.5 Determine the distribution of rooted super maps of the complete bipartite graphs $K_(m,n)$, $m, n \geq 3$, by automorphism group orders.

E14.6 Determine the distribution of rooted super maps of super cube Q_n, $n \geq 4$, by automorphism group orders.

E14.7 Determine the distribution of rooted super maps of the complete tripartite graph K_l, m, n, $l, m, n \geq 2$, by automorphism group orders.

E14.8 Determine the distribution of rooted super maps of the complete equi-bipartite $K_{n,n}$, $n \geq 3$, by automorphism group orders.

E14.9 Determine the distribution of rooted super maps of the complete equi-tripartite $K_(n, n, n)$, $n \geq 2$, by automorphism group orders.

E14.10 Determine the distribution of rooted super maps of the complete quadpartite graph $K_{k,l,m,n}$, $k, l, m, n \geq 2$, by automorphism group orders.

E14.11 Determine the distribution of rooted super maps of the complete equi-quadpartite graph $K_{n,n,n,n}$, $n \geq 2$, by automorphism group orders.

E14.12 Determine the distribution of rooted super maps of super wheel W_n, $n \geq 4$, by automorphism group orders.

XIV.8 Researches

R14.1 For given integer $n \geq 1$, determine the distribution of outer planar graphs of order n by the order of their semi-automorphism groups.

R14.2 For given integer $n \geq 1$, determine the distribution of Eulerian planar graphs of order n by the order of their semi-automorphism groups.

R14.3 For given integer $n \geq 1$, determine the distribution of general planar graphs of order n by the order of their semi-automorphism groups.

R14.4 For given integer $n \geq 1$, determine the distribution of nonseparable planar graphs of order n by the order of their semi-automorphism groups.

R14.5 For given integer $n \geq 1$, determine the distribution of cubic planar graphs of order n by the order of their semi-automorphism groups.

R14.6 For given integer $n \geq 1$, determine the distribution

of 4-regular planar graphs of order n by the order of their semi-automorphism groups.

R14.7 For given integer $n \geq 1$, determine the distribution of nonseparable graphs of order n by the order of their semi-automorphism groups.

R14.8 For given integer $n \geq 1$, determine the distribution of general graphs of order n by the order of their semi-automorphism groups.

R14.9 For given integer $n \geq 1$, determine the distribution of Elerian graphs of order n by the order of their semi-automorphism groups.

R14.10 For given integer $n \geq 1$, determine the distribution of general graphs of order n by the order of their semi-automorphism groups.

R14.11 For given integer $n \geq 1$, determine the distribution of cubic graphs of order n by the order of their semi-automorphism groups.

R14.12 For given integer $n \geq 1$, determine the distribution of 4-regular graphs of order n by the order of their semi-automorphism groups.

R14.13 For given integer $n \geq 1$, determine the distribution of 5-regular graphs of order n by the order of their semi-automorphism groups.

Chapter XV

Equations with Partitions

- The meson functional is used for describing equations discovered from census of maps via vertex, or face, partition as parameters.
- Functional equations are extracted form the census of general maps and nonseparable maps with the root-vertex valency and the vertex partition vector on the sphere.
- By observing maps without cut-edge on general surfaces, a functional equation has also be found with vertex partition.
- Functional equations satisfied by the vertex partition functions of Eulerian maps on the sphere and general surfaces are derived from suitable decompositions of related sets of maps.
- All these equations can be shown to be well definedness. However, they are not yet solved in any way.

XV.1 The meson functional

Let $f(\underline{y}) \in \mathcal{R}\{\underline{y}\}$, where $\underline{y} = (y_1, y_2, \cdots)$, be a function, and

$$V(f, y_i) \geq 0,\ i = 1, 2, \cdots.$$

A transformation is established as $\int_y : \quad y^i \quad \mapsto \quad y_i,\ i = 1, , 2, \cdots,$ convinced $y^0 = 1 \mapsto y_0$.

Since $\displaystyle\int_y$ is a function from the function space $\boldsymbol{\mathcal{F}}$ with basis $\{1, y, y^2, \cdots\}$ to the vector space $\boldsymbol{\mathcal{V}}$ with basis $\{y_0, y_1, y_2, \cdots\}$, it is called the *meson functional. i.e.*, the *Blissard operator*. For any

$$v_i = \sum_{j\geq 0} a_{ij} y^j,\ i = 1, 2,$$

it is easy to check that

$$\begin{aligned}\int_y (v_1 + v_2) &= \sum_{j\geq 0} (a_{1j} + a_{2j}) \int_y y^j \\ &= \sum_{j\geq 0} a_{1j} y_j + \sum_{j\geq 0} a_{2j} y_j \\ &= \int_y v_1 + \int_y v_2.\end{aligned}$$

Hence, the meson functional is linear.

The inverse of the meson functional $\displaystyle\int_y$ is denoted by $\displaystyle\int_y^{-1}\ :\ y_j \mapsto y^j,\ i = 1, 2, \cdots$, convinced $\displaystyle\int_y^{-1} y_0 = 1$, or simply $y_0 = 1$. However, 1 is seen as a vector in $\boldsymbol{\mathcal{V}}$.

Two linear operators called *left* an d*right projection*, denoted by, respectively, $\Im_y$ and $\Re_y$, are defined in the space $\boldsymbol{\mathcal{V}}$ as: let $v = \sum_{j\geq 0} a_j y_j \in \boldsymbol{\mathcal{V}}$, then

$$\begin{cases} \Im_y v = \displaystyle\sum_{j\geq 0} (j+1) a_{j+1} y_j; \\ \Re_y v = \displaystyle\sum_{j\geq 1} \frac{1}{j} a_{j-1} y_j. \end{cases} \tag{15.1}$$

In other words, if y_i is considered as the vector with all entries 0 but only the i-th 1, then the matrices corresponding to $\Im_y$ and $\Re_y$ are, respectively, as

$$L = (\underline{l}_1^{\mathrm{T}}, \underline{l}_2^{\mathrm{T}}, \underline{l}_3^{\mathrm{T}}, \cdots) \tag{15.2a}$$

where

$$\underline{l}_j = \begin{cases} \underline{0}, \text{ when } j=1; \\ (j-1)\underline{1}_{j-1}, \text{ when } j \geq 2 \end{cases}$$

for $\underline{1}_j$ being the infinite vector of all entries 0 but only the $(j-1)$-st 1 and

$$R^{\mathrm{T}} = (\underline{r}_1^{\mathrm{T}}, \underline{r}_2^{\mathrm{T}}, \underline{r}_3^{\mathrm{T}}, \cdots) \tag{15.2b}$$

where

$$\underline{r}_j = \begin{cases} \underline{0}, \text{ when } j=1; \\ \dfrac{1}{j-1}\underline{1}_{j-1}, \text{ when } j \geq 2 \end{cases}$$

for the super index 'T' as the transpose.

Easy to check that

$$LR = \begin{pmatrix} I & \underline{0}^{\mathrm{T}} \\ \underline{0} & 0 \end{pmatrix}, \ RL = \begin{pmatrix} 0 & \underline{0} \\ \underline{0}^{\mathrm{T}} & I \end{pmatrix}, \tag{15.3}$$

where I is the identity.

Theorem 15.1 For $v = v(y_0, y_1, \cdots) \in \boldsymbol{\mathcal{V}}$, let $f(y) = \displaystyle\int_y^{-1} v$, then

$$\frac{\mathrm{d}}{\mathrm{d}y} f(y) = \int_y^{-1} \Im_y v; \ \int f(y)\mathrm{d}y = \int_y^{-1} \Re_y v. \tag{15.4}$$

Proof By equating the coefficients of terms in same type on the two sides, the theorem is done. □

If $f(\underline{x}, \underline{y})$ is a function with two types of unknowns, and assume $f(\underline{x}, \underline{y}) \in \boldsymbol{\mathcal{V}}(\underline{x}, \underline{y})$, a bilinear space, then it is easily checked that

$$\int_x^{-1}\int_y^{-1} f(\underline{x}, \underline{y}) = \int_y^{-1}\int_x^{-1} f(\underline{x}, \underline{y}). \tag{15.5}$$

Denoted by $F(\underline{x}, \underline{y})$ the function in (15.5). Conversely, for $F(x, y) \in \mathcal{R}(x, y)$, we have

$$f(\underline{x}, \underline{y}) = \int_x \int_y F(x, y) \tag{15.6}$$

because of interchangeable between $\displaystyle\int_x$ and $\displaystyle\int_y$.

Let $f(z) \in \mathcal{R}\{z\}$. The following two operators on f as

$$\delta_{x,y} f = \frac{f(x) - f(y)}{x - y}, \tag{15.7}$$

and

$$\partial_{x,y} f = \frac{y f(x) - x f(y)}{x - y}. \tag{15.8}$$

are, respectively, called the (x, y)-*difference* and $\langle x, y\rangle$-*difference* of f with respect to z.

Lemma 15.1 For any function $f(z) \in \mathcal{R}\{x\}$, let $f = f(z)$, then

$$\partial_{x,y}(zf) = xy\delta_{x,y} f. \tag{15.9}$$

Proof Because of the linearity of the two operators $\partial_{x,y}$ and $\delta_{x,y}$, this enables us only to discuss $f(z) = z^n,\ n > 0$. Then, it is seen

$$\begin{aligned}\partial_{x,y} z f &= \partial_{x,y} z^{n+1} \\ &= \frac{y x^{n+1} - x y^{n+1}}{x - y} \\ &= xy \frac{x^n - y^n}{x - y} \\ &= xy \delta_{x,y} z^n \\ &= xy \delta_{x,y} f.\end{aligned}$$

This is what we want to prove. □

Theorem 15.2 For any $f \in \mathcal{R}\{z\}$, we have

$$x^2 y^2 \delta^2_{x^2,y^2}(zf) - \partial^2_{x^2,y^2}(zf) = x^2 y^2 \delta_{x^2,y^2}(zf^2). \tag{15.10}$$

Proof From (15.7) and (15.8), the left hand side of (15.10) is

$$\begin{aligned}&\frac{x^2y^2((x^2f(x^2) - y^2f(y^2))^2 - x^2y^2(f(x^2) - f(y^2))^2)}{x^2 - y^2} \\ &= \frac{x^2y^2(x^2f^2(x^2) - y^2f^2(y^2))}{x^2 - y^2}.\end{aligned}$$

From (15.7), this is the right hand side of (15.10). □

For a set of maps $\mathcal{A}$, let

$$f_{\mathcal{A}}(x, \underline{y}) = \sum_{A \in \mathcal{A}} x^{m(A)} \underline{y}^{\underline{n}(A)} \tag{15.11}$$

where $m(A)$ and $\underline{n}(A)$ are, respectively, the invariant parameter and vector on $\mathcal{A}$. Let $F_{\mathcal{A}}(x, y)$ be such a function of two unknowns that

$$f_{\mathcal{A}}(x, \underline{y}) = \int_y F_{\mathcal{A}}(x, y). \tag{15.12}$$

The powers of x and y in $F_{\mathcal{A}}(x, y)$ are, respectively, called the *first parameter* and the *second parameter*.

Theorem 15.3 Let $\mathcal{S}$ and $\mathcal{T}$ be two sets of maps. If there is a mapping $\lambda(T) = \{S_1, S_2, \cdots, S_{m(T)+1}\}$ such that S_i and $\{i, m(T)+2-i\}$ are with a 1–1 correspondence from $\mathcal{T}$ to $\mathcal{S}$ for any $T \in \mathcal{T}$, where i and $m(T)+2-i$ are the contributions to, respectively, the first and the second parameters, $i = 1, 2, \cdots, m(T)+1$, with the condition as

$$\mathcal{S} = \sum_{T \in \mathcal{T}} \langle(T),$$

then

$$F_{\mathcal{S}}(x, y) = xy\delta_{x,y}(zf_{\mathcal{T}}) \tag{15.13}$$

where $f_{\mathcal{T}} = f_{\mathcal{T}}(z) = f_{\mathcal{T}}(z, \underline{y})$.

Proof From the definition of λ, we have

$$\begin{aligned}
F_{\mathcal{S}}(x, y) &= \sum_{T \in \mathcal{T}} \sum_{i=1}^{m(T)+1} x^i y^{m(T)-i+2} \underline{y}^{\underline{n}(T)} \\
&= xy \sum_{T \in \mathcal{T}} \frac{x^{m(T)+1} - y^{m(T)+1}}{x - y} \underline{y}^{\underline{n}(T)} \\
&= xy\delta_{x,y}(zf_{\mathcal{T}}).
\end{aligned}$$

This is (15.13). $\square$

Theorem 15.4 Let $\mathcal{S}$ and $\mathcal{T}$ be two sets of maps. If there exists a mapping $\lambda(T) = \{S_1, S_2, \cdots, S_{m(T)-1}\}$ such that S_i and $\{i, m(T)-i\}$

are in a 1–1 correspondence for $T \in \mathcal{T}$, where i and $m(T)+2-i$ are the contributions to, respectively, the first and the second parameters, $i = 1, 2, \cdots, m(T)-1$, with the condition

$$\mathcal{S} = \sum_{T\in\mathcal{T}} \langle(T),$$

then

$$F_{\mathcal{S}}(x,y) = \partial_{x,y}(f_{\mathcal{T}}) \tag{15.14}$$

where $f_{\mathcal{T}} = f_{\mathcal{T}}(z) = f_{\mathcal{T}}(z, \underline{y})$.

Proof From the definition of λ, we have

$$\begin{aligned} F_{\mathcal{S}}(x,y) &= \sum_{T\in\mathcal{T}} \sum_{i=1}^{m(T)-1} x^i y^{m(T)-i} \underline{y}^{\underline{n}(T)} \\ &= xy \sum_{T\in\mathcal{T}} \frac{yx^{m(T)} - xy^{m(T)}}{x-y} \underline{y}^{\underline{n}(T)} \\ &= \partial_{x,y}(f_{\mathcal{T}}). \end{aligned}$$

This is (15.14). □

XV.2 General maps on the sphere

A map is said to be *general* if both loops and multi-edges are allowed. Of course, the vertex map ϑ is also treated as degenerate. Let $\mathcal{M}_{\text{gep}}$ be the set of all rooted general planar maps. For any $M \in \mathcal{M}_{\text{gep}}$, let $a = e_r(M)$ be the root-edge. Then, $\mathcal{M}_{\text{gep}}$ can be divided into three classes: $\mathcal{M}_{\text{gep}_0}$, $\mathcal{M}_{\text{gep}_1}$, and $\mathcal{M}_{\text{gep}_2}$, *i.e.*,

$$\mathcal{M}_{\text{gep}} = \mathcal{M}_{\text{gep}_0} + \mathcal{M}_{\text{gep}_1} + \mathcal{M}_{\text{gep}_2} \tag{15.15}$$

such that $\mathcal{M}_{\text{gep}_0}$ consists of a single map ϑ,

$$\mathcal{M}_{\text{gep}_1} = \{M | \forall M \in \mathcal{M}_{\text{gep}}, a \text{ is a loop}\}.$$

Of course,

$$\mathcal{M}_{\text{gep}_2} = \{M | \forall M \in \mathcal{M}_{\text{gep}}, a \text{ is a link}\}$$

in its own right.

Lemma 15.2 Let $\mathcal{M}_{\langle \text{gep} \rangle_1} = \{M - a | \forall M \in \mathcal{M}_{\text{gep}_1}\}$. Then, we have

$$\mathcal{M}_{\langle \text{gep} \rangle_1} = \mathcal{M}_{\text{gep}} \bigodot \mathcal{M}_{\text{gep}} \tag{15.16}$$

where $\odot$ is the 1-production as defined in §2.1.

Proof For a map $M \in \mathcal{M}_{\langle \text{gep} \rangle_1}$, because there is a map $\tilde{M} \in \mathcal{M}_{\langle \text{gep} \rangle_1}$, such that $M = \tilde{M} - \tilde{a}, \tilde{a} = e_r(\tilde{M})$, the root-edge of $\tilde{M}$, by considering the root-edge $\tilde{a}$ as a loop we see that $M = M_1 \dot{+} M_2$, provided $M_1 \cap M_2 = o$, the common root-vertex of M and $\tilde{M}$. Since M_1 and M_2 are allowed to be any maps in $\mathcal{M}_{\text{gep}}$ including the vertex map ϑ, this implies that $M \in \mathcal{M}_{\text{gep}} \odot \mathcal{M}_{\text{gep}}$.

Conversely, for any $M \in \mathcal{M}_{\text{gep}} \odot \mathcal{M}_{\text{gep}}$, since $M = M_1 \dot{+} M_2$, M_1, $M_2 \in \mathcal{M}_{\text{gep}}$, we may always construct a map $\tilde{M}$ by adding a loop $\tilde{a}$ at the common vertex of M_1 and M_2 as the root-edge of $\tilde{M}$ such that M_1 and M_2 are in different domains of the loop. Of course, $\tilde{M}$ is a general map. Because the root-edge of $\tilde{M}$ is a loop added, $\tilde{M} \in \mathcal{M}_{\text{gep}_1}$. However, it is easily seen that $M = \tilde{M} - \tilde{a}$. Therefore, $M \in \mathcal{M}_{\langle \text{gep} \rangle_1}$.

In consequence, the lemma is proved. □

For $\mathcal{M}_{\text{gep}_2}$, because the root-edges are all links we consider the set $\mathcal{M}_{(\text{gep})_2} = \{M \bullet a | \forall M \in \mathcal{M}_{\text{gep}_2}\}$, $a = e_r(M)$, the root-edge as usual. The smallest map in $\mathcal{M}_{\text{gep}_2}$ is the link map $L = (Kr, (r)(\alpha\beta r))$ and it is seen that $L \bullet a = \vartheta$. Thus, $\vartheta \in \mathcal{M}_{(\text{gep})_2}$. For any $M \in \mathcal{M}_{\text{gep}_2}$, because the root-edge of M is not a loop we know that $M \bullet a \in \mathcal{M}_{\text{gep}}$. Conversely, for any $M \in \mathcal{M}_{\text{gep}}$ we may always construct a map $\tilde{M} \in \mathcal{M}_{\text{gep}_2}$ by splitting the root-vertex of $\tilde{M}$ into two vertices with a new edge $\tilde{a}$ as the root-edge connecting them. This implies that $\tilde{M} \bullet \tilde{a} = M \in \mathcal{M}_{(\text{gep})_2}$. Therefore we have

$$\mathcal{M}_{(\text{gep})_2} = \mathcal{M}_{\text{gep}}. \tag{15.17}$$

Lemma 15.3 For $\mathcal{M}_{\text{gep}_2}$ we have

$$\mathcal{M}_{\text{gep}_2} = \sum_{M \in \mathcal{M}_{\text{gep}}} \{\nabla_i M | \ 0 \leq i \leq m(M)\} \tag{15.18}$$

where $m(M)$ is the valency of the root-vertex of M and ∇_i is the operator defined in §7.1.

Proof For any $M \in \mathcal{M}_{\text{gep}_2}$, because the root-edge a is a link, we may assume $a = (o_1, o_2)$ such that

$$o_1 = (r, S) \text{ and } o_2 = (\alpha\beta r, T).$$

Let $\tilde{M}$ be the map obtained by contracting the root-edge a into a vertex $\tilde{o} = (T, S)$ as the root-vertex of $\tilde{M}$. It is easily checked that $\tilde{M} \in \mathcal{M}_{\text{gep}}$ from (15.17) and that

$$M = \nabla_{|S|}\tilde{M}, \ 0 \leq |S| \leq m(\tilde{M})$$

where $m(\tilde{M}) = |S| + |T|$, and $|Z|, Z = S$ or T, stands for the cardinality of Z. That implies M is a member of the set on the right hand side of (15.18).

Conversely, for any M in the set on the right, because there exist a map $\tilde{M} \in \mathcal{M}_{\text{gep}}$ and an integer $i, 0 \leq i \leq m(\tilde{M})$, such that $M = \nabla_i\tilde{M}$, we may soon find that $M \in \mathcal{M}_{\text{gep}_2}$ by considering that the root-edge of M is always a link and that $M \in \mathcal{M}_{\text{gep}}$ as well. Thus, $M \in \mathcal{M}_{\text{gep}_2}$.

Therefore the lemma follows. □

From the two Lemmas above we are now allowed to determine the contributions of $\mathcal{M}_{\text{gep}_i}, i = 0, 1, 2$, to the enufunction

$$g_{\mathcal{M}_{\text{gep}}}(x, \underline{y}) = \sum_{M \in \mathcal{M}_{\text{gep}}} x^{m(M)} \underline{y}^{\underline{n}(M)} \tag{15.19}$$

where $\underline{n}(M) = (n_1(M), n_2(M), \cdots, n_i(M), \cdots)$, $n_i(m)$ is the number of vertices of valency i in M and $m(M)$, the valency of the root-vertex of M.

First, since ϑ has neither non-rooted vertex nor edge we soon see that

$$g_{\mathcal{M}_{\mathrm{gep}_0}} = 1. \tag{15.20}$$

Then, by Lemma 15.3,

$$g_{\mathcal{M}_{\mathrm{gep}_1}} = x^2 g^2 \tag{15.21}$$

where $g = g_{\mathcal{M}_{\mathrm{gep}}}(x, \underline{y})$ defined by (15.19).

Further, from Lemma 15.3

$$g_{\mathcal{M}_{\mathrm{gep}_2}} = \int_y \sum_{M \in \mathcal{M}_{\mathrm{gep}}} \left(\sum_{i=1}^{m(M)+1} x^i y^{m(M)-i+2} \right) \underline{y}^{\underline{n}(M)}.$$

By Theorem 15.3,

$$g_{\mathcal{M}_{\mathrm{gep}_2}} = x \int_y (y \delta_{x,y}(zg)). \tag{15.22}$$

Theorem 15.5 The enufunction g defined by (15..5) satisfies the following functional equation:

$$g = 1 + x^2 g^2 + x \int_y (y \delta_{x,y}(zg)). \tag{15.23}$$

Proof According to (15.15), from (15.20–22) the theorem is soon obtained. □

XV.3 Nonseparable maps on the sphere

Let $\mathcal{M}_{\mathrm{ns}}$ be the set of all rooted nonseparable planar maps with the convention that the loop map $L_1 = (Kr, (r, \alpha\beta r))$ is included but the link map $L = (Kr, (r)(\alpha\beta r))$ is not for convenience.

Then, $\mathcal{M}_{\mathrm{ns}}$ is divided into two parts $\mathcal{M}_{\mathrm{ns}_0}$ and $\mathcal{M}_{\mathrm{ns}_1}$, *i.e.*,

$$\mathcal{M}_{\mathrm{ns}} = \mathcal{M}_{\mathrm{ns}_0} + \mathcal{M}_{\mathrm{ns}_1} \tag{15.24}$$

such that $\mathcal{M}_{\mathrm{ns}_0}$ consists of only the loop map L_1.

Lemma 15.4 A map $M \in \mathcal{M}_{\rm ns}$, $M \neq L_1$ if, and only if, its dual $M^* \in \mathcal{M}_{\rm ns}$.

Proof By contradiction. Assume $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P}) \in \mathcal{M}_{\rm ns}$, $M \neq L_1$ and its dual $M^* = (\mathcal{X}_{\beta,\alpha}, \mathcal{P}\alpha\beta) \notin \mathcal{M}_{\rm ns}$. Let

$$v^* = (x, \mathcal{P}\alpha\beta x, \cdots, (\mathcal{P}\alpha\beta)^m x)$$

be a cut-vertex of M^*. Then we have a face $f^* = (x, \mathcal{P}x, \cdots, \mathcal{P}^n x)$ on M^* such that there exists an integer $j, 1 \leq j \leq n$, on f^* satisfying $\mathcal{P}^j x = (\mathcal{P}\alpha\beta)^i x$ for some $i, 1 \leq i \leq m$, *i.e.*, $v_x^* = v_{\mathcal{P}^j x}^* = v^*$. However, f^* is a vertex of M which has the face v^* having the symmetry and hence f^* is a cut-vertex of M. A contradiction to the assumption appears. The necessity is true.

Conversely, from the duality the sufficiency is true as well. □

For any $M \in \mathcal{M}_{\rm ns}$, let $m(M)$ be the valency of the root-vertex and $\underline{n}(M) = (n_1(M), n_2(M), \cdots), n_i(M)$ be the number of nonrooted vertices of valency $i, i \geq 1$.

From the nonimputability, the root-edge $a = (v_1, v_{\beta r})$ of any map M in $\mathcal{M}_{\rm ns_1}$ is always a link. The map $M \bullet a$ obtained by contracting the root-edge a in M has the same number of faces as M does.

Lemma 15.5 For any $M \in \mathcal{M}_{\rm ns_1}$ there is an integer $k \geq 1$ with

$$M \bullet a = \sum_{i=1}^{k} \cdot M_i \tag{15.25}$$

such that all M_i are allowed to be any map in $\mathcal{M}_{\rm ns}$ and that $M_i, i = 1, 2, \cdots, k$, does not have the form (15.25) for $k > 1$.

Proof In fact, from what were mentioned in §6.2, we see that k is the root-index of M and that all $M_i, 1 \leq i \leq k$, do not have the form (15.2) for $k > 1$. From the nonseparability of M, by considering that all vertices of M_i except for the root-vertex are the same as those of M for $i = 1, 2, \cdots, k$, since M_i does not have the form (15.2) for

$k > 1$, the root-vertex is not a cut-vertex for $i = 1, 2, \cdots, k$. That implies all $M_i, 1 \le i \le k$, are allowed to be any map in $\mathcal{M}_{\mathrm{ns}}$ including the loop map. The lemma follows. □

Now let us write

$$\mathcal{M}_k = \{\sum_{i=1}^{k} \cdot M_i | \ \forall M_i \in \mathcal{M}_{\mathrm{ns}}, 1 \le i \le k\}, \tag{15.26}$$

and

$$\mathcal{M}_{(\mathrm{ns})_1} = \{M \bullet a | \forall M \in \mathcal{M}_{\mathrm{ns}_1}\}, \tag{15.27}$$

where $a = e_r$, the root-edge of M.

Lemma 15.6 For $\mathcal{M}_{\mathrm{ns}_1}$, we have

$$\mathcal{M}_{(\mathrm{ns})_1} = \sum_{k \ge 1} \mathcal{M}_k;$$

$$\mathcal{M}_k = \mathcal{M}_{\mathrm{ns}}^{\times k} \tag{15.28}$$

where $\times$ is the inner $1v$-production.

Proof By the definition of inner $1v$-product, the last form of (15.28) is easily seen.

From Lemma 15.5 we can find that

$$\mathcal{M}_{(\mathrm{ns})_1} = \bigcup_{k \ge 1} \mathcal{M}_k.$$

Moreover, for any $i, j, i \ne j$, we always have

$$\mathcal{M}_i \bigcap \mathcal{M}_j = \emptyset.$$

Therefore, the first form of (15.28) is true. □

Based on the two lemmas above, we are allowed to evaluate the contributions of $\mathcal{M}_{\mathrm{ns}_0}$ and $\mathcal{M}_{\mathrm{ns}_1}$ to the enufunction $f_{\mathcal{M}_{\mathrm{ns}}}$ of $\mathcal{M}_{\mathrm{ns}}$ with vertex partition, *i.e.*,

$$f = f_{\mathcal{M}_{\mathrm{ns}}} = \sum_{M \in \mathcal{M}_{\mathrm{ns}}} x^{m(M)} \underline{y}^{\underline{n}(M)}, \tag{15.29}$$

where $m(M)$ is the valency of root-vertex and

$$\underline{n}(M) = (n_1(M), n_2(M), \cdots)$$

with $n_i(M)$ being the number of nonroot-vertices of valency $i, i \geq 1$.

Since $\mathcal{M}_{\mathrm{ns}_0}$ consists of only the loop map, which has the root-vertex of valency 2 without nonrooted vertex, we have

$$f_{\mathcal{M}_{\mathrm{ns}_0}} = x^2. \tag{15.30}$$

For $\mathcal{M}_{\mathrm{ns}_1}$, we have to evaluate the function

$$\tilde{f}(x,z) = \sum_{M \in \mathcal{M}_{\mathrm{ns}_1}} x^{m(M)} z^{s(M)} \underline{y}^{\underline{n}(M)} \tag{15.31}$$

where $s(M)$ is the valency of the nonrooted vertex $v_{\beta r}$ incident with the root-edge e_r of M.

By considering that for $M \in \mathcal{M}_{\mathrm{ns}_1}$, $m(M \bullet a) = (m(M) - 1) + (s(M) - 1)$, we soon find

$$\tilde{f}(x,z) = xz \sum_{\tilde{M} \in \mathcal{M}_{\mathrm{ns}_1}} x^{m(\tilde{M}) - s(\tilde{M})} z^{s(\tilde{M})} \underline{y}^{\underline{n}(\tilde{M})}$$

where $s(\tilde{M})$ is the contribution of the valency of the nonrooted end of the root-edge of M to the valency of the root-vertex of $\tilde{M} = M \bullet a$, $M \in \mathcal{M}_{\mathrm{ns}_1}$. Because $s(\tilde{M})$ is allowed to be any number between 1 and $m(\tilde{M}) - 1$, from Lemma 15.2 we have

$$\tilde{f}(x,z) = xz \sum_{k \geq 1} \left(\sum_{M \in \mathcal{M}_{\mathrm{ns}}} x^{m(M)} \sum_{i=1}^{m(M)-1} (\frac{z}{x})^i \underline{y}^{\underline{n}(M)} \right)^k.$$

By Theorem 15.4,

$$\begin{aligned} \tilde{f}(x,z) &= xz \sum_{k \geq 1} (\partial_{x,z} f)^k \\ &= \frac{xz \partial_{x,z} f}{1 - \partial_{x,z} f}, \end{aligned}$$

where $f = f(u) = f_{\mathcal{M}_{\mathrm{ns}}}(u, \underline{y})$, and hence

$$f_{\mathcal{M}_{\mathrm{ns}_1}} = \int_y \tilde{f}(x,y)$$

$$= x \int_y \frac{y\partial_{x,y} f}{1 - \partial_{x,y} f}\,. \tag{15.32}$$

Theorem 15.6 The enufunction of $\mathcal{M}_{\rm ns}$ defined by (15.29) satisfies the following functional equation:

$$f = x^2 + x \int_y \frac{y\partial_{x,y} f}{1 - \partial_{x,y} f}\,. \tag{15.33}$$

Proof Since $f_{\mathcal{M}_{\rm ns}} = f_{\mathcal{M}_{{\rm ns}_0}} + f_{\mathcal{M}_{{\rm ns}_1}}$, from (15.30) and (15.32) the theorem is obtained. □

XV.4 Maps without cut-edge on surfaces

In this section, only maps without cut-edge(or, 2-connected)are considered. Let $\mathcal{M}$ be the set of all(including both orientable and nonorientable) rooted maps without cut-edge. Classify $\mathcal{M}$ into three classes as

$$\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1 + \mathcal{M}_2 \tag{15.34}$$

where $\mathcal{M}_0$ consists of only the vertex map ϑ, $\mathcal{M}_1$ is of all with the root-edge self-loop and, of course, $\mathcal{M}_2$ is of all with the root-edge not self-loop.

Lemma 15.7 The contribution of the set $\mathcal{M}_0$ to $f = f_{\mathcal{M}}(x.\underline{y})$ is

$$f_0 = 1, \tag{15.35}$$

where $f_0 = f_{\mathcal{M}_0}(x, \underline{y})$.

Proof Because of ϑ neither cut-edge nor nonrooted vertex, $m(\vartheta) = 0$ and $\underline{n}(\vartheta) = \underline{0}$. Thus, the lemma is obtained. □

In order to determine the enufunction of $\mathcal{M}_1$, how to decompose $\mathcal{M}_1$ should first be considered.

Lemma 15.8 For $\mathcal{M}_1$, we have

$$\mathcal{M}_{\langle 1\rangle} = \mathcal{M}, \tag{15.36}$$

where $\mathcal{M}_{\langle 1\rangle} = \{M - a|\forall M \in \mathcal{M}_1\}$, $a = Kr(M)$.

Proof Because of $L_1 = (r, \gamma r) \in \mathcal{M}_1$, $\gamma = \alpha\beta$, we have $L_1 - a = \vartheta \in \mathcal{M}$.

For any $S \in \mathcal{M}_{\langle 1\rangle}$, since there exists $M \in \mathcal{M}$ such that $S = M - a$, by considering the root-edge of M not cut-edge, it is seen $S \in \mathcal{M}$. Thus, $\mathcal{M}_{\langle 1\rangle} \subseteq \mathcal{M}$.

Conversely, for any $M = (\mathcal{X}, \mathcal{J}) \in \mathcal{M}$, a new edge $a' = Kr'$ is added to the root-vertex $(r)_{\mathcal{J}}$ for getting S_i, whose root-vertex is

$$(r'r, \cdots, \mathcal{J}^i r, \gamma r', \mathcal{J}^{i+1} r, \cdots, \mathcal{J}^{m(M)-1} r),$$

where $0 \leq i \leq m(M) - 1$. Because of $S_i - a' = M$, we have $S_i \in \mathcal{M}_1$. Hence, $\mathcal{M} \subseteq \mathcal{M}_{\langle 1\rangle}$. □

From this lemma, it is seen that each map $M = (\mathcal{X}, \mathcal{J})$ in $\mathcal{M}$ not only produces $S_i \in \mathcal{M}_1, 0 \leq i \leq m(M) - 1$ but also $S_m \in \mathcal{M}_1$ nonisomorphic to them. Its root-vertex is $(r', \langle r\rangle_{\mathcal{J}}, \gamma r')$. For $M \in \mathcal{M}$, let

$$\mathcal{S}_M = \{S_i | 0 \leq i \leq m(M)\}. \tag{15.37}$$

Lemma 15.9 The set $\mathcal{M}_1$ has a decomposition as

$$\mathcal{M}_1 = \sum_{M \in \mathcal{M}} \mathcal{S}_M, \tag{15.38}$$

where $\mathcal{S}_M$ is given from (15.37).

Proof First, for $M \in \mathcal{M}_1$, because of $M' = M - a \in \mathcal{M}_{\langle 1\rangle}$, Lemma 15.8 enables us to have $M' \in \mathcal{M}$. Via (15.37), $M \in \mathcal{S}_{M'}$ is obtained. Thus, what on the left hand side of (7.4.5) is a subset of that on the right hand side.

Conversely, for a map M on the left hand side of (15.38), because of the root-edge a self-loop,we have $M \in \mathcal{M}_1$. Thus, the set on the left hand side of (15.38) is a subset of that on the right hand side. □

On the basis of this lemma, we have

Lemma 15.10 For $g_1 = g_{\mathcal{M}^1}(x.\underline{y}) = f_{\mathcal{M}^1}(x^2, \underline{y})$, we have

$$f_1 = x^2(f + \frac{\partial f}{\partial x}), \tag{15.39}$$

where $f = f_{\mathcal{M}}(x, \underline{y})$.

Proof From Lemma 15.9,

$$f_1 = \sum_{M \in \mathcal{M}} (m(M) + 1) x^{m(M)} \underline{y}^{\underline{n}}.$$

By Lemma 9.10, we get

$$f_1 = x^2 (f + x \frac{\partial f}{\partial x}).$$

This is the conclusion of the lemma. □

In what follows, $\mathcal{M}_2$ is considered.

Lemma 15.11 For $\mathcal{M}_2$, let $\mathcal{M}_{(2)} = \{M \bullet a | \forall M \in \mathcal{M}_2\}$, then

$$\mathcal{M}_{(2)} = \mathcal{M} - \vartheta, \tag{15.40}$$

where ϑ is the vertex map.

Proof For any $M \in \mathcal{M}_{(2)}$, there is a map $M' \in \mathcal{M}_2$ such that $M = M' \bullet a'$. Because of a' neither cut-edge nor self-loop, $M \in \mathcal{M}$. And, sice the link map $L_0 = (Kr, (r)(\gamma r)) \notin \mathcal{M}_2$, $\mathcal{M}_{(2)} \subseteq \mathcal{M} - \vartheta$.

Conversely, for any $M = (\mathcal{X}, \mathcal{J}) \in \mathcal{M} - \vartheta$, let U_{i+1} be obtained by splitting the root-vertex $(r)_{\mathcal{J}}$ of M with an additional edge $a' = Kr'$ whose two ends are $(r', r, \cdots, \mathcal{J}^i r)$ and

$$(\gamma r', \mathcal{J}^{i+1} r, \cdots, \mathcal{J}^{m(M)-1} r), \ 1 \leq i \leq m(M).$$

Because of a' not cut-edge, $U_i \in \mathcal{M}_2$, $1 \leq i \leq m(M)$. And, because of $M = U_i \bullet a'$, $M \in \mathcal{M}_{(2)}$. Thus, $\mathcal{M} \subseteq \vartheta \subseteq \mathcal{M}_{(2)}$. □

For any $M = (\mathcal{X}, \mathcal{J}) \in \mathcal{M} - \vartheta$, let

$$\mathcal{U}_M = \{U_i | 1 \leq i \leq m(M)\}, \tag{15.41}$$

where U_i is appeared in the proof of Lemma 15.11.

Lemma 15.12 The set $\mathcal{M}^2$ has the following decomposition:

$$\mathcal{M}_2 = \sum_{M \in \mathcal{M} - \vartheta} \mathcal{U}_M, \tag{15.42}$$

where $\mathcal{U}_M$ is given from (15.41).

Proof First, for any $M \in \mathcal{M}_2$, from Lemma 15.5, $M' = M \bullet a \in \mathcal{M} - \vartheta$ and further $M \in \mathcal{U}_{M'}$. This implies that

$$\mathcal{M}_2 = \bigcup_{M \in \mathcal{M} - \vartheta} \mathcal{U}_M.$$

Then, for any b$M_1, M_2 \in \mathcal{M} - \vartheta$, because of M_1 not isomorphic to M_2,

$$\mathcal{U}_{M_1} \bigcap \mathcal{U}_{M_2} = \emptyset.$$

Thus, (15.42) is right. The lemma is obtained. □

This lemma enables us to determine the contribution of $\mathcal{M}_2$ to $f_{\mathcal{M}}(x, \underline{y})$.

Lemma 15.13 For $f_2 = f_{\mathcal{M}_2}(x, \underline{y})$, we have

$$f_2 = x \int_y y \partial_{x,y} f \tag{15.43}$$

where $f = f(z) = f_{\mathcal{M}}(z, \underline{y})$.

Proof From Lemma 15.12,

$$f_2 = \int_y \sum_{M \in \mathcal{M} - \vartheta} \left(\sum_{i=1}^{m(M)} x^{i+1} y^{m(M)+2-i} \right) \underline{y}^{\underline{n}(M)}.$$

By employing Theorem 15.4, (15.43) is obtained. □

On the basis of those having been done, the main result can be deduced in what follows.

Theorem 15.7 The functional equation about f

$$x^2 \frac{\partial f}{\partial x} = -1 + (1 - x^2) f - x \int_y y \partial_{x,y} f \tag{15.44}$$

is well defined on the field $\mathcal{L}\{\Re; x, \underline{y}\}$. And, its solution is $f = f(x) = f_{\mathcal{M}}(x, \underline{y})$.

Proof The first statement can be proved in a usual way except for involving a certain complication.

The second statement is derived from (15.34) in companion with (15.35), (15.39), and (15.43). □

XV.5 Eulerian maps on the sphere

A map is called *Eulerian* if all the valencies of its vertices are even (or say, all vertices are *even*). Let $\mathcal{U}$ be the set of all the rooted planar Eulerian maps with the convention that the vertex map ϑ is in $\mathcal{U}$ for convenience.

Further, $\mathcal{U}$ is divided into 3 classes: $\mathcal{U}_0$, $\mathcal{U}_1$ and $\mathcal{U}_2$, *i.e.*,

$$\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1 + \mathcal{U}_2 \tag{15.45}$$

such that $\mathcal{U}_0 = \{\vartheta\}$, or simply write $\{\vartheta\} = \vartheta$, and

$$\mathcal{U}_1 = \{U | \forall U \in \mathcal{U} \text{ with } a = e_r(U) \text{ being a loop}\}.$$

Lemma 15.14 Any Eulerian map (not necessarily planar) has no cut-edge.

Proof By contradiction. Assume that a Eulerian map M has a cut-edge $e = (u, v)$ such that $M = M_1 \cup e \cup M_2$, $M_1 \cap M_2 = \emptyset$, where M_1 and M_2 are submaps of M with the property that M_1 is incident to u and M_2, to v. From the Eulerianity of M, u and v are the unique odd vertex in M_1 and M_2, respectively. This contradicts to that both M_1 and M_2 are a submap of M because the number of odd vertices in a map is even. □

Lemma 15.15 Let $\mathcal{U}_{\langle 1\rangle} = \{U - a | \forall U \in \mathcal{U}_1\}$ where $a = e_r(U)$ is the root-edge. Then, we have

$$\mathcal{U}_{\langle 1\rangle} = \mathcal{U} \bigodot \mathcal{U} \tag{15.46}$$

where $\odot$ is the $1v$-production.

Proof Because for $U \in \mathcal{U}_1$, the root-edge a is a loop, we see that $U - a = U_1 \dot{+} U_2$ where U_1 and U_2 are in the inner and outer domain of a respectively. Of course, it can be checked that both U_1 and U_2 are maps in $\mathcal{U}$. Thus, the set on the left hand side of (15.46) is a subset of that on the right.

On the other hand, for any $U = U_1 \dot{+} U_2,\ U_1, U_2 \in \mathcal{U}$, we may uniquely construct a map U' by adding a loop at the common vertex of U_1 and U_2. The root-edge of U' is chosen to be the loop such that U_1 and U_2 are respectively in its inner and outer domains. It is easily checked that U' is a Eulerian map and hence $U' \in \mathcal{U}_1$. However, $U = U' - a \in \mathcal{U}_{\langle 1 \rangle}$. That implies the set on the right hand side of (15.46) is a subset of that on the left as well. □

For any map $U \in \mathcal{U}_2$, we see that the root-edge a of U has to be a link. From Lemma 15.14, if $U \bullet a = U_1 \dot{+} U_2$ such that the root-vertex is the common vertex, then the valencies of the vertices in both U_1 and U_2 are odd. Further for any $U \in \mathcal{U}$, if $U = U_1 \dot{+} U_2$, then the valencies of the common vertex between U_1 and U_2 are both even as well.

Lemma 15.16 Let $\mathcal{U}_{(2)} = \{U \bullet a | \forall U \in \mathcal{U}_2\}$ where a is the root-edge of U. Then, we have

$$\mathcal{U}_{(2)} = \mathcal{U} - \vartheta \tag{15.47}$$

where $\vartheta = \mathcal{U}_0$ for simplicity.

Proof From what has just been discussed, the set on the left hand side of (15.47) is a subset of that on the right.

Conversely for any $U \in \mathcal{U} - \vartheta$, we may always construct a map U' by splitting the root-vertex into o_1 and o_2 with the new edge $a' = (o_1, o_2)$ as the root-edge of U' such that the valencies of o_1 and o_2 in U' are both even. However, $U = U' \bullet a' \in \mathcal{U}_{(2)}$. That implies the set on the right hand side of (15.47) is a subset of that on the left. □

For a map $U \in \mathcal{U} - \vartheta$, assume the valency of the root-vertex o is $2k,\ k \geq 1$,without loss of generality. The map U' obtained by

splitting the root-vertex into o_1 and o_2 with the new edge $a' = (o_1, o_2)$ such that the valency $\rho(o_1; U') = 2i$ and hence $\rho(o_2; U') = 2k - 2i + 2$ is denoted by $U_{[2i]}, i = 1, 2, \cdots, k$.

From Lemma 15.14, we see that the procedure works and that all the resultant maps $U_{[2i]}$ are also Eulerian maps.

Lemma 15.17 For $\mathcal{U}_2$, we have

$$|\mathcal{U}_2| = \sum_{U \in \mathcal{U} - \vartheta} |\{U_{[2i]}|\ i = 1, 2, \cdots, m(U)\}| \tag{15.48}$$

where $2m(U)$ is the valency of the root-vertex of U.

Proof First, we show that for any $U \in \mathcal{U}_2$, it appears in the set on the right hand side of (15.48) only once. Assume that $a = (o, v)$ is the root-edge of U and that $\rho(o) = 2s$ and $\rho(v) = 2t$. Let U' be the map obtained by contracting the root-edge a, *i.e.*, $U' = U \bullet a$. Then, there is the only possibility that $U = U'_{[2s]}$ in the set on the right hand side of (15.48).

Then, we show that for any map U in the set on the right hand side of (15.48), it appears also only once in $\mathcal{U}_2$. This is obvious from Lemma 15.16 because all elements are distinguished and they are all maps in $\mathcal{U}_2$ by considering the Eulerianity with the root-edges being links. $\square$

In what follows, we see what kind of equation should be satisfied by the enufunction u of rooted planar Eulerian maps with vertex partition. Write

$$u = \sum_{U \in \mathcal{U}} x^{2m(U)} \underline{y}^{\underline{n}(U)} \tag{15.49}$$

where $2m(U)$ is the valency of the root-vertex as mentioned above and $\underline{n}(U) = (n_2(U), \cdots, n_{2i}(U), \cdots), n_{2i}(U)$ is the number of nonrooted vertices of valency $2i$, $i \geq 1$.

Theorem 15.8 The function u defined in (15.5) satisfies the

following functional equation:

$$u = 1 + x^2u^2 + x^2 \int_y \Big(y^2 \delta_{x^2,y^2}(u(\sqrt{z})) \Big) \tag{15.50}$$

where $u(z) = u|_{x=z} = u(z, \underline{y})$.

Proof The contribution of $\mathcal{U}_0$ to u is

$$u_0 = 1 \tag{15.51}$$

since $2m(\vartheta) = 0$ and $\underline{n}(\vartheta) = \underline{0}$.

From Lemma 15.15, the contribution of $\mathcal{U}_1$ to u is

$$\begin{aligned} u_1 &= x^2 \sum_{U \in \mathcal{U}_{\langle 1 \rangle}} x^{2m(U)} \underline{y}^{\underline{n}(U)} \\ &= x^2u^2. \end{aligned} \tag{15.52}$$

The contribution of $\mathcal{U}_2$ to u is denoted by u_2. Let

$$\tilde{u}(z) = \sum_{U \in \mathcal{U}_2} x^{2m(U)} z^{2j(U)} \underline{y}^{\underline{\tilde{n}}(U)}$$

where $2j(U)$ is the valency of the nonrooted end of the root-edge and $\underline{\tilde{n}}(U) = (\tilde{n}_2(U), \tilde{n}_4(U), \cdots, \tilde{n}_{2i}(U), \cdots)$, $\tilde{n}_{2i}(U)$ is the number of vertices of valency $2i$ except for the two ends of the root-edge. It is easily seen that

$$\underline{\tilde{n}}(U) = \underline{n}(U) - e_{2j(U)}$$

where $e_{2j(U)}$ is the vector with all the components 0 except only for the $j(U)$-th which is 1. In addition, it can be verified that

$$u_2 = \int_y \tilde{u}(y). \tag{15.53}$$

From Lemma 15.17, we have

$$\tilde{u}(z) = \sum_{U \in \mathcal{U} - \vartheta} \left(\sum_{i=1}^{m(U)} x^{2i} z^{2m(U)-2i+2} \right) \underline{y}^{\underline{n}(U)}.$$

By Theorem 15.3,

$$\tilde{u}(z) = x^2 z^2 \delta_{z^2,x^2}(u(\sqrt{t})).$$

Then by (15.53), we find that

$$u_2 = x^2 \int_y y^2 \delta_{y^2,x^2}(u(\sqrt{t})). \tag{15.54}$$

From (15.51), (15.52) and (15.54), the theorem is obtained. $\square$

XV.6 Eulerian maps on the surfaces

Let $\mathcal{M}_{\mathrm{Eul}}$ be the set of all orientable Euler rooted maps on surfaces. Because of no cut-edge for any Eulerian map, Eulerian maps are classified into three classes as $\mathcal{M}^0_{\mathrm{Eul}}$, $\mathcal{M}^1_{\mathrm{Eul}}$ and $\mathcal{M}^2_{\mathrm{Eul}}$ such that $\mathcal{M}^0_{\mathrm{Eul}}$ consists of only the vertex map ϑ, $\mathcal{M}^1_{\mathrm{Eul}}$ has all its maps with the root-edge self-loop and

$$\mathcal{M}^2_{\mathrm{Eul}} = \mathcal{M}_{\mathrm{Eul}} - \mathcal{M}^0_{\mathrm{Eul}} - \mathcal{M}^1_{\mathrm{Eul}}. \tag{15.55}$$

Naturally, $\mathcal{M}^2_{\mathrm{Eul}}$ has all maps with the root-edge a link.

The enufunction $g = f_{\mathcal{M}_{\mathrm{Eul}}}(x, \underline{y})$ is of the powers $2m$ and $\underline{n} = (n_2, n_4, \cdots)$ of, respectively, x and $\underline{y}$ as the valency of root-vertex and the the vertex partition operator.

Lemma 15.18 For $\mathcal{M}^0_{\mathrm{Eul}}$, we have

$$g_0 = 1, \tag{15.56}$$

where $g_0 = f_{\mathcal{M}^0_{\mathrm{Eul}}}(x^2, \underline{y})$.

Proof Because of ϑ with neither root-edge nor nonrooted vertex, $m(\vartheta) = 0$ and $\underline{n}(\vartheta) = \underline{0}$. The lemma is done. $\square$

In order to determine the enufunction of $\mathcal{M}^1_{\mathrm{Eul}}$, a suitable decomposition of $\mathcal{M}^1_{\mathrm{Eul}}$ should be first considered.

Lemma 15.19 For $\mathcal{M}^1_{\mathrm{Eul}}$, we have

$$\mathcal{M}^{\langle 1 \rangle}_{\mathrm{Eul}} = \mathcal{M}_{\mathrm{Eul}}, \tag{15.57}$$

where $\mathcal{M}_{\rm Eul}^{\langle 1\rangle} = \{M - a|\forall M \in \mathcal{M}_{\rm Eul}^1\}$, $a = Kr(M)$, the root-edge.

Proof Because of $L_1 = (r, \gamma r) \in \mathcal{M}_{\rm Eul}^1$, $\gamma\alpha\beta$, $L_1 - a = \vartheta \in \mathcal{M}_{\rm Eul}$ is seen.

For any $S \in \mathcal{M}_{\rm Eul}^{\langle}1\rangle$, since there is a map $M \in \mathcal{M}_{\rm Eul}$ such that $S = M - a$, from M as a Eulerian map, $S \in \mathcal{M}_{\rm Eul}$ is known. Thus, $\mathcal{M}_{\rm Eul}^{\langle 1\rangle} \subseteq \mathcal{M}_{\rm Eul}$.

Conversely, for any $M = (\mathcal{X}, \mathcal{J}) \in \mathcal{M}_{\rm Eul}$, By adding a new root-edge $a' = Kr'$ at the vertex $(r)_{\mathcal{J}}$ to get S_i, whose root-vertex is $(r'r, \cdots, \mathcal{J}^i r, \gamma r', \mathcal{J}^{i+1} r, \cdots, \mathcal{J}^{2m(M)-1} r)$, $0 \leq i \leq 2m(M) - 1$. From $S_i - a' = M$, $S_i \in \mathcal{M}_{\rm Eul}^1$. Thus, $\mathcal{M}_{\rm Eul} \subseteq \mathcal{M}_{\rm Eul}^{\langle 1\rangle}$. □

In the proof of this lemma, it is seen that each map $M = (\mathcal{X}, \mathcal{J})$ in $\mathcal{M}_{\rm Eul}$ produces not only $S_i \in \mathcal{M}_{\rm Eul}^1, 0 \leq i \leq 2m(M) - 1$, but also $S_{2m} \in \mathcal{M}_{\rm Eul}^1$ nonisomorphic to them. Its root-vertex is $(r', \langle r\rangle_{\mathcal{J}}, \gamma r')$.

For $M \in \mathcal{M}_{\rm Eul}$, let

$$\mathcal{S}_M = \{S_i | 0 \leq i \leq 2m(M)\}. \tag{15.58}$$

Lemma 15.20 Set $\mathcal{M}_{\rm Eul}^1$ has the following decomposition:

$$\mathcal{M}_{\rm Eul}^1 = \sum_{M \in \mathcal{M}_{\rm Eul}} \mathcal{S}_M, \tag{15.59}$$

where $\mathcal{S}_M$ is given from (15.58).

Proof First, for any $M \in \mathcal{M}_{\rm Eul}^1$, from Lemma 15.19, $M' = M - a \in \mathcal{M}_{\rm Eul}$, and hence $M \in \mathcal{S}_{M'}$. Thus,

$$\mathcal{M}_{\rm Eul}^1 = \bigcup_{M \in \mathcal{M}_{\rm Eul}} \mathcal{S}_M.$$

Then, for any M_1, $M_2 \in \mathcal{M}_{\rm Eul}$, because of nonisomorphic between them,

$$\mathcal{S}_{M_1} \bigcap \mathcal{S}_{M_2} = \emptyset.$$

Therefore, the conclusion of the lemma is true. □

On the basis of this lemma, the following lemma can be seen.

Lemma 15.21 For $g_1 = g\mathcal{M}^1_{\mathrm{Eul}}(x.\underline{y}) = f_{\mathcal{M}^1_{\mathrm{Eul}}}(x^2, \underline{y})$, we have

$$g_1 = x^2(g + 2x^2\frac{\partial g}{\partial x^2}), \tag{15.60}$$

where $g = g_{\mathcal{M}_{\mathrm{Eul}}}(x, \underline{y}) = f_{\mathcal{M}_{\mathrm{Eul}}}(x^2, \underline{y})$.

Proof From (15.59),

$$g_1 = \sum_{M\in\mathcal{M}_{\mathrm{Eul}}} (2m(M) + 1)x^{m(M)}\underline{y}^{\underline{n}}.$$

By employing Lemma 9.10, (15.60) is obtained. □

In what follows, $\mathcal{M}^2_{\mathrm{Eul}}$ is investigated.

Lemma 15.22 For $\mathcal{M}^2_{\mathrm{Eul}}$, let $\mathcal{M}^{(2)}_{\mathrm{Eul}} = \{M \bullet a|\forall M \in \mathcal{M}^2_{\mathrm{Eul}}\}$, then

$$\mathcal{M}^{(2)}_{\mathrm{Eul}} = \mathcal{M}_{\mathrm{Eul}} - \vartheta, \tag{15.61}$$

where ϑ is the vertex map.

Proof Because of $L_1 \notin \mathcal{M}^2_{\mathrm{Eul}}$, $\vartheta \notin \mathcal{M}^{(}_{\mathrm{Eul}}(2))$. Then, $\mathcal{M}^2_{\mathrm{Eul}} \subseteq \mathcal{M}_{\mathrm{Eul}} - \vartheta$.

Conversely, for any $M = (\mathcal{X}, \mathcal{P}) \in \mathcal{M}_{\mathrm{Eul}} - \vartheta$, since $M_{2j} = (\mathcal{X} + Kr_{2j}, \mathcal{P}_{2j}) \in \mathcal{M}^2_{\mathrm{Eul}}$ where the two ends of $a_{2j} = Kr_{2j}$ is obtained by splitting the root-vertex $(r)_{\mathcal{P}}$ of M, *i.e.*,

$$(r_{2j})_{\mathcal{P}_{2j}} = (r_{2j}, r, \mathcal{P}r, \cdots, (\mathcal{P})^{2j-2})$$

and

$$(\gamma r_{2j})_{\mathcal{P}_{2j}} = (\gamma r_{2j}, \mathcal{P})^{2j-1}r, \cdots, (\mathcal{P})^{2m-1}),$$

$1 \leq i \leq m - 1$. Because of $M = M_{2j} \bullet a_{2j}$, $M_{2j} \in \mathcal{M}^2_{\mathrm{Eul}}$. Thus, $\mathcal{M}_{\mathrm{Eul}} - \vartheta \subseteq \mathcal{M}^2_{\mathrm{Eul}}$. □

For any $M = (\mathcal{X}, \mathcal{J}) \in \mathcal{M}_{\mathrm{Eul}} - \vartheta$, let

$$\mathcal{M}_M = \{M_{2j}|1 \leq j \leq m(M)\}, \tag{15.62}$$

where M_{2j}, $1 \leq j \leq m(M) - 1$, have appeared in the proof of Lemma 15.22.

Lemma 15.23 Set ${\mathcal{M}_{\mathrm{Eul}}}^2$ has the following decomposition:

$$
{\mathcal{M}_{\mathrm{Eul}}}^2 = \sum_{M \in \mathcal{M}_{\mathrm{Eul}} - \vartheta} \mathcal{M}_M \tag{15.63}
$$

where $\mathcal{M}_M$ is given from (15.62).

Proof First, for any $M \in \mathcal{M}_{\mathrm{Eul}}^2$, because of $M' = M \bullet a \in \mathcal{M}_{\mathrm{Eul}} - \vartheta$, Lemma 15.22 tells us that $M \in \mathcal{M}_(M)$. Thus,

$$
{\mathcal{M}_{\mathrm{Eul}}}^2 = \bigcup_{M \in \mathcal{M}_{\mathrm{Eul}} - \vartheta} \mathcal{M}_M.
$$

Then, for any $M_1, M_2 \in \mathcal{M}_{\mathrm{Eul}} - \vartheta$, because of nonisomorphic between M_1 and M_2,

$$
\mathcal{M}_{M_1} \bigcap \mathcal{M}_{M_2} \neq \emptyset.
$$

This implies (15.63). □

On the basis of this lemma, the following conclusion can be seen.

Lemma 15.24 For $g_2 = f_{\mathcal{M}_{\mathrm{Eul}}^2}(x, \underline{y})$, we have

$$
g_2 = x^2 \int_y y^2 \delta_{x^2, y^2} g(\sqrt{z}) \tag{15.64}
$$

where $g = g(x) = f_{\mathcal{M}_{\mathrm{Eul}}}(x^2, \underline{y})$.

Proof From Lemma 15.23,

$$
g_2 = \sum_{M \in \mathcal{M}_{\mathrm{Eul}} - \vartheta} \int_y \Big(\sum_{j=1}^{m(M)} x^{2j} y^{2m(M)+2-2j} \Big) \underline{y}^{\underline{n}}.
$$

By employing Theorem 15.3,

$$
g_2 = x^2 \int_y y^2 \delta_{x^2, y^2} g(\sqrt{z}).
$$

This is the lemma. □

Now, the main result of this section can be described.

Theorem 15.9 The functional equation about g

$$2x^4\frac{\partial g}{\partial x^2} = -1 + (1 - x^2)g - x^2 \int_y \delta_{x^2,y^2} g(\sqrt{z}) \tag{15.65}$$

is well defined on the field $\mathcal{L}\{\Re; x, \underline{y}\}$. Further, its solution is $g = g_{\mathcal{M}_{\mathrm{Eul}}}(x, \underline{y}) = f_{\mathcal{M}_{\mathrm{Eul}}}(x, \underline{y})$.

Proof The last conclusion is deduced from (15.55), in companion with (15.56), (15.60), and (15.64).

The former conclusion is a result of the well definedness for the equation system obtained by equating the coefficients on the two sides of (15.65). $\square$

Activities on Chapter XV

XV.7 Observations

O15.1 Let $f = f_{\mathcal{U}}(x.\underline{y})$ be the vertex partition function of a set $\mathcal{U}$ of maps. Suppose $\mathcal{U}_+ = \{U + a' | \forall U \in \mathcal{U}\}$ where $a' = Kr'$ is the root-edge of $U' = U + a'$. To evaluate $f_+ = f_{\mathcal{U}_+}(x, \underline{y})$ from f.

O15.2 Let $\mathcal{A}$ be the set of supermaps of K_4, the complete graph of order 4. To evaluate $f_{\mathcal{A}}(x, \underline{y})$.

O15.3 Let $\mathcal{B}_1$ be the set of all bipartite rooted maps with the root-edge a cut-edge and $\mathcal{B}\langle 1 \rangle = \{B - a | \forall B \in \mathcal{B}_1\}$. Suppose the vertex partition function f of all bipartite rooted maps on surfaces is known. To evaluate $f_{\mathcal{B}_1}(x, \underline{y})$ from f.

O15.4 For a set of maps $\mathcal{M}$, observe the relationship between vertex partition function $f_{\mathcal{M}}(x, \underline{y})$ and the enufunction $f_{\mathcal{M}}(x, y)$ of two parameters: the valency of root-vertex and the size.

O15.5 For a set of maps $\mathcal{M}$, observe the relationship between vertex partition function $f_{\mathcal{M}}(x, \underline{y})$ and the enufunction $f_{\mathcal{M}}(x, y)$ of two parameters: the valency of root-vertex and the order.

O15.6 For a set of maps $\mathcal{M}$, observe the relationship between vertex partition function $f_{\mathcal{M}}(x, \underline{y})$ and the enufunction $f_{\mathcal{M}}(x, y)$ of two parameters: the valency of root-vertex and the coorder.

O15.7 Consider the conditions satisfied by the face partition of petal bundle with size $n \geq 1$.

O15.8 Consider the conditions satisfied by the face partition of a supermap of the complete graph K_n with order $n \geq 4$.

O15.9 Show a 1-to-1 correspondence between the two sets $\mathcal{U}'$ and $\mathcal{M}$ where $\mathcal{U}' = \{U - a | \forall U = (\mathcal{X}, \mathcal{J}) \in \mathcal{U}, a = Kr(U), \gamma r \in (r)_{\mathcal{J}}, r = r(U)\}$, $\mathcal{U}$ consists of all rooted maps on the projective plane and $\mathcal{M}$ is of all rooted planar maps.

O15.10 Observe the relationship between the vertex partition function of plane rooted trees and the face partition function of outerplanar rooted maps.

O15.11 Observe the existence of a tree for a given vector $\underline{n}$ as its vertex partition.

XV.8 Exercises

E15.1 Prove that the vertex partition function of planted trees satisfies the functional equation about f as

$$y_1 = (1 - \int_y \frac{y^2}{1 - yf})f. \tag{15.66}$$

E15.2 Solve the functional equation (15.66) in a direct way.

E15.3 If the vertex partition function of Halin rooted maps is taken to have the root-face valency instead of root-vertex valency, then show that the function satisfies the functional equation about f as

$$f = x^2 y_3 + \frac{x}{f} \int_y \frac{y^2 f_y^2}{x - y f_y} \tag{15.67}$$

where $f = f(x) = f(x, \underline{y})$ and $f_y = f(y)$.

E15.4 Provide a method for solving the functional equation shown in (15.67).

E15.5 Prove that the vertex partition function of Wintersweets with root not on a circuit satisfies the following functional equation about f as

$$(1 - \frac{x y_3}{1 - y_2}) f = 1 + x \int_y y \delta_{x,y}(z f_z). \tag{15.68}$$

E15.6 Find a way for solving the functional equation shown in (15.68).

E15.7 Prove that the vertex partition function of unicyclic maps with root not on the circuit satisfies the functional equation about f as

$$f = x^2\tau_1 + x\int_y y\partial_{x,y} f_z \tag{15.69}$$

where τ_1 is the vertex partition function of planted trees, which is known in E15.1.

E15.8 Solve the functional equation shown in (15.69).

E15.9 Show that the following functional equation about f is satisfied by the vertex partition function of outerplanar rooted maps as

$$(1 - x^2\varphi)f = 1 + x\int_y y\delta_{x,y}(zf_z) \tag{15.70}$$

where

$$\varphi = \frac{1}{2x^2}(1 - \sqrt{1 - 4x^2}).$$

E15.10 Solve the functional equation shown in (15.70).

E15.11 Find a functional equation satisfied by the vertex partition function of general planar rooted maps.

XV.9 Researches

R15.1 For given orientable genus $p \neq 0$, determine a functional equation satisfied by the vertex function of a set of maps on the surface of genus p.

R15.2 For given nonorientable genus $q \geq 1$, determine a functional equation satisfied by the vertex function of a set of maps on the surface of genus q.

R15.3 Determine a functional equation satisfied by the vertex partition function of nonseparable rooted maps on the Klein bottle.

R15.4 Determine a functional equation satisfied by the vertex partition function of nonseparable rooted maps on the torus.

R15.5 Determine a functional equation satisfied by the vertex partition function of bipartite rooted maps on the torus.

R15.6 Solve the functional equation about f as

$$(1 - x^2 f)f = 1 + x \int_y y\delta_{x,y}(zf_z). \tag{15.71}$$

R15.7 Solve the functional equation about f as

$$(xf + \int_y f_y).f = \int_y (f_y + y^2\delta x^2, y^2 f_{\sqrt{z}}). \tag{15.72}$$

R15.8 Solve the functional equation about f as

$$(\int_y (1 + xy)f_y - x^2 f)f = \int_y (f_y + xy\delta_{x,y}(zf_z)). \tag{15.73}$$

R15.9 Solve the functional equation about f as

$$f = \int_y \frac{1}{1 - \partial_{x,y}(z^2)f_z}. \tag{15.74}$$

.

R15.10 Solve the functional equation about f as

$$f = \int_y \frac{1 - \partial_{x^2,y^2}(zf_{\sqrt{z}})}{1 - 2\partial_{x^2,y^2}(zf_{\sqrt{z}}) - x^2y^2\delta_{x^2,y^2}(zf^2_{\sqrt{z}})}. \tag{15.75}$$

R15.11 Solve the functional equation about f as

$$f = x^2 + x \int_y \frac{y\partial_{x,y} f_z}{1 - \partial_{x,y} f_z}. \tag{15.76}$$

R15.12 Solve the functional equation about f as

$$f = x^2 + x^2 \int_y \frac{y^2\delta_{x^2,y^2} f_{\sqrt{z}}}{(1 - \partial_{x^2,y^2} f_{\sqrt{z}})^2 - (xy\delta_{x^2,y^2} f_{\sqrt{z}})^2}. \tag{15.77}$$

R15.13 Solve the functional equation about f as

$$(1 - x^2 f)f = 1 + x^2 \int_y y^2\delta_{x^2,y^2} f_{\sqrt{z}}. \tag{15.78}$$

Appendix I

Concepts of Polyhedra, Surfaces, Embeddings and Maps

This appendix provides a fundamental of basic concepts of polyhedra, surfaces, embeddings and maps from original to developed as a compensation for Chapters I–II. Only those available in the usage from combinatorization to algebraication are particularly concentrated on.

Ax.I.1 Polyhedra

A *polyhedron* P is a set $\{C_i|1 \leq i \leq k\}$, $k \geq 1$, of cycles of letters such that each letter occurs exactly twice with the same power (or index) or different powers: 1(always omitted) and -1 and denoted by $P = (\{C_i|1 \leq i \leq k\})$. It is seen as a set of all the cycles in any cyclic order.

This is a general statement of Heffter's[Hef1] (and more than half a century later Edmonds'[Edm1] as dual case) which has the minimality of no proper subset as a polyhedron for the convenience usages.

A polyhedron is *orientable* if there is an orientation of each cycle, clockwise or anticlockwise, such that the two occurrences of each letter with different powers; *nonorientable*, otherwise.

The *support* of polyhedron $P = (\{C_i|1 \leq i \leq k\})$ is the graph $G = (V_P, E_P)$ with a weight w on E_P where $V_P = \{C_i|1 \leq i \leq k\}$, $(C_i, C_j) \in E_P$ if, and only if, C_i and C_j, $1 \leq i, j \leq k$, have a common letter, and

$$w(e) = \begin{cases} 0, & \text{when two powers are different;} \\ 1, & \text{otherwise} \end{cases} \tag{I.1}$$

for $e \in E_P$.

The set of all the edges with weight 1 is called the 1-*set* of the polyhedron.

Theorem I.1 A polyhedron $P = (\{C_i | 1 \leq i \leq k\})$ is orientable if, and only if, one of the following statements is satisfied:

(1) What obtained by contracting all edges of weight 0 on the support is a bipartite graph;

(2) No odd weight fundamental circuit is on the support;

(3) No odd weight circuit is on the support;

(4) The 1-set forms a cocycle;

(5) The equation system about $x_i = x_{C_i}$, $C_i \in V_P$, on GF(2)

$$x_i + x_j = w(C_i, C_j) \tag{I.2}$$

for $(C_i, C_j) \in E_P$ has a solution.

Proof Because P is orientable, the two occurrences of each letter are with different powers. Since the weights of all edges are the constant 0, the equation system (I.2) has a solution of $x_i = 0$ for all $C_i \in V_P$, $1 \leq i \leq k$. Further, by considering that the consistency of equation system (I.2) is not changed from switching the orientation of a cycle between ce clockwise and anticlockwise while interchanging the weights between 0 and 1 of all the edges incident with the cycle on the support, statement (5) is satisfied for any orientable polyhedron.

On the basis of statement (5), from a solution of equation system (I.2) the vertices of G_P are classified into two classes by $x_i = 1$ or 0: 1-class or 0-class respectively. According to equation (I.2), each edge with weight 1 has its two ends in different classes and hence the 1-set is a cocycle. This is statement (4).

On the basis of statement (4), since any circuit meets even number of edges with a cocycle, all circuits are with even weight. This means no odd weight circuit. Therefore, statement (3) is satisfied.

On the basis of statement (3), the statement (2) is naturally deduced because a fundamental circuit is a circuit in its own right.

On the basis of statement (2), by contracting all edges of weight 0 in each fundamental circuit on the support, (1) is satisfied.

On the basis of statement (1),the vertices are partitioned into two classes by the equivalence that two vertices are joined by even weight path. By switching the orientation of all vertices in one of the two classes and those in the other class unchanged, a polyhedron without weight 1 edge is found. This implies that P is orientable.

In summary, the theorem is proved. □

On the support $G_P = (V_P, E_P)$ of a polyhedron P, the operation of switching the orientations of all vertices in a subset of V_P between clockwise and anticlockwise and the weights of all edges incident with just one end in the subset interchanged between 0 and 1 is called a *switch* on P.

Theorem I.2 The orientability of a polyhedron does not change under switches.

Proof From the definition of orientability, the conclusion of this theorem is true. □

Let T be a minimal set of edges having an edge in common with all cocycles in the support of a polyhedron. In fact, it can seen that T is a spanning tree.

All the polyhedra obtained by switching on a polyhedron P are seen to be the *same* as P; *different*, otherwise. From Theorem I.2, in order to discuss all different polyhedra it enables us only to consider all such polyhedra of the support with weight 0 on all tree edge for a spanning tree chosen independently in any convenient way. Such a polyhedron is said to be *classic*.

Theorem I.3 A classic polyhedron is orientable if, and only if, all edges as letters have their two occurrences with different powers. A classic polyhedron is nonorientable if, and only if, the set of letters each of which has its two occurrences with same power does not contain a cocycle.

Proof The first statement is deduced from Theorem I.1(3). The second statement is by contradiction derived from Theorem I.1 and Theorem I.2. □

Now, a polyhedron(always summed to be classic below) P is considered as a permutation formed by its cycles. Let δ be the permutation with each cycle only consists of the two occurrences of each letter in P. Then, the *dual*, denoted by P^*, of P is defined to be $P^* = P\delta$ such that their supports are with the same weight. The cycles in P are called *faces* and those in P^* are *vertices*. Cycles in δ are edges. Let $\nu(P)$, $\epsilon(P)$ and $\phi(P)$ be, respectively, the number of vertices, edges and faces on P, then $\nu(P) - \epsilon(P) + \phi(P)$ is the *Eulerian characteristic* of P. The graph which is formed by vertices and edges of P is called a *skeleton* of P. Of course, the skeleton of P is the support of P^*.

Theorem I.4 P^* is a polyhedron and $P^{**} = P$. P^* is orientable if, and only if, so is P with the same Eulerian characteristic.

Proof It is easily checked that P^* is a polyhedron from P as a polyhedron. Since $\delta^2 = 1$, the identity, we have

$$P^{**} = P^*\delta = (P\delta)\delta = P(\delta^2) = P.$$

This is the first statement. From Theorem I.3, the second statement is obtained. □

Ax.I.2 Surfaces

Surfaces seen as polyhedral polygons can be topologically classified by a type of equivalence. Let $\mathbf{P}$ be the set of all such polygons.

For $P = (\{(A_i)|i \geq 1\}) \in \mathbf{P}$, the following three operations including their inverses are called *elementary transformation*:

Operation 0: For $(A_i) = (Xaa^{-1}Y)$, $(A_i) \Leftrightarrow (XY)$ where at least one of X and Y is not empty;

Operation 1: For $(A_i) = (XabYab)$(or $(XabYb^{-1}a^{-1})$), $(A_i) \Leftrightarrow (XaYa)$(or $XaYa^{-1}$);

Operation 2: For $(A_i) = (Xa)$ and $(A_j) = (a^{-1}Y)$, $i \neq j$, $(\{(A_i), (A_j)\}) \Leftrightarrow (XY)$ where at least one of X and Y is not empty. Particularly, $(\{(A_i), (A_j)\}) \Leftrightarrow (XaYa^{-1})$ when both (X) and (Y) are polyhedra.

If a polyhedron P can be obtained by elementary transformation into another polyhedron Q, then they are called *elementary equivalence*, denoted by $P \sim_{\text{el}} Q$. In topology, the elementary equivalence is topological in 2-dimensional sense.

Lemma I.1 For $P \in \mathbf{P}$, there exists a polyhedron $Q = (X) \in \mathbf{P}$ where X is a linear order such that $P \sim_{\text{el}} Q$.

Proof Let $P = (\{(A_i) | 1 \leq i \leq k\})$. If $k = 1$, P is in the form as Q itself. If $k \geq 2$, by employing Operation 2 step by step to reduce the number of cycles 1 by 1 if any, the form Q can be found. $\square$

Lemma I.2 For $P \in \mathbf{P}$, if $P = ((A)(B))$ with both (A) and (B) as polyhedra, then for any $x \notin A \cup B$, $P \sim_{\text{el}} ((A)x(B)x^{-1})$.

Proof It is seen that

$$\begin{aligned} P = (AB) &\sim_{\text{el}} (Axx^{-1}B) \text{ (by Operation 0)} \\ &\sim_{\text{el}} ((Ax)(x^{-1}B)) \text{ (by Operation 2)} \\ &= ((A)x(B)x^{-1}). \end{aligned}$$

$\square$

From Lemmas I.1–2, for classifying $\mathbf{P}$ it suffices to only discuss polygons as Q.

Lemma I.3 Let $Q = (AxByCx^{-1}Dy^{-1})$, then

$$Q \sim_{\text{el}} (ADxyBx^{-1}Cy^{-1}). \tag{I.3}$$

Proof It is seen that

$$\begin{aligned} Q &\sim_{\text{el}} ((Axz)(z^{-1}ByCx^{-1}Dy^{-1})) \text{ (by Operation 2)} \\ &\sim_{\text{el}} (zADy^{-1}z^{-1}ByC) \text{ (by Operation 2)} \\ &= (ADxyBx^{-1}Cy^{-1}). \end{aligned}$$

$\square$

Lemma I.4 Let $Q = (AxByCx^{-1}Dy^{-1})$, then

$$Q \sim_{\mathrm{el}} (BAxyx^{-1}DCy^{-1}). \tag{I.4}$$

Proof It is seen that

$$\begin{aligned} Q &\sim_{\mathrm{el}} ((x^{-1}Dy^{-1}Axz)(ByCz^{-1})) \text{ (by Operation 2)} \\ &\sim_{\mathrm{el}} (BAxzx^{-1}DCz^{-1}) \text{ (by Operation 2)} \\ &= (BAxyx^{-1}DCy^{-1}). \end{aligned}$$ □

Lemma I.5 Let $Q = (AxByCx^{-1}Dy^{-1})$, then

$$Q \sim_{\mathrm{el}} (ADCBxyx^{-1}y^{-1}). \tag{I.5}$$

Proof From Lemma I.4 and then Lemma I.3, the lemma is soon done. □

According to Lemma I.5, if A is replaced by EA in polyhedron $(ADCB)$, then the relation is soon derived as

Relation 1: $(AxByCx^{-1}Dy^{-1}E) \sim_{\mathrm{el}} (ADCBExyx^{-1}y^{-1})$.

Lemma I.6 Let $Q = (AxBx) \in \mathbf{P}$, then $Q \sim_{\mathrm{el}} (AB^{-1}xx)$.

Proof It is seen that

$$\begin{aligned} Q &\sim_{\mathrm{el}} ((Axz)(z^{-1}Bx)) = ((zAx)(x^{-1}B^{-1}z)) \text{ (by Operation 2)} \\ &\sim_{\mathrm{el}} (zAB^{-1}z) = (AB^{-1}xx) \text{ (by Operation 2)}. \end{aligned}$$ □

According to Lemma I.6, if A is replaced by CA in polyhedron (AB^{-1}), then the relation is soon derived as

Relation 2: $(AxBxC) \sim_{\mathrm{el}} (AB^{-1}Cxx)$.

Lemma I.7 Let$Q = (Axyx^{-1}y^{-1}zz) \in \mathbf{P}$, then$Q \sim_{\mathrm{el}} (Axyzyxz)$.

Proof It is seen that

$$\begin{aligned} Q &\sim_{\mathrm{el}} ((zAxyt)(t^{-1}x^{-1}y^{-1}z)) \text{ (by Operation 2)} \\ &\sim_{\mathrm{el}} (Axytyxt) \text{ (by Operation 2)} \\ &= (Axyzyxz). \end{aligned}$$ □

According to Lemma I.7, then by Relation 2 twice for x and y, the relation is soon derived as

Relation 3: $(Axyx^{-1}y^{-1}zz) \sim_{\mathrm{el}} (Axxyyzz)$.

Lemma I.8 If $Q \in \mathbf{P}$ orientable not as $(AxByCx^{-1}Dy^{-1}E)$, then $Q \sim_{\mathrm{el}} (xx^{-1})(= O_0)$.

Proof Because Q is not in the above form, Q has to be in form as $(Axx^{-1}B)$. If both A and B are empty, them $Q \sim_{\mathrm{el}} (xx^{-1})$; otherwise, $Q \sim_{\mathrm{el}} (AB)$. Because (AB) still satisfies the given condition, by the finite recursion principle, (xx^{-1}) can be found. □

Theorem I.5 For any $Q \in \mathbf{P}$ orientable, if $Q \not\sim_{\mathrm{el}} (xx^{-1})$, then there exists an integer $p \geq 1$ such that

$$Q \sim_{\mathrm{el}} (\prod_{i=1}^{p} x_i y_i x_i^{-1} y_i^{-1})(= O_p). \tag{I.5}$$

Proof Because $Q \sim_{\mathrm{el}} (Ax_1By_1Cx_1^{-1}Dy_1^{-1}E)$, by Relation 1 we have

$$Q \sim_{\mathrm{el}} (ADCBEx_1y_1x_1^{-1}y_1^{-1}).$$

If $(ADCBE) \sim_{\mathrm{el}} (xx^{-1})$, the $Q \sim_{\mathrm{el}} (x_1y_1x_1^{-1}y_1^{-1})$. That is the case $p = 1$. Otherwise $(ADCBE) = (A_1x_2B_1y_2C_1x_2^{-1}D_1y_2^{-1}E_1)$. Because

$$(ADCBEx_1y_1x_1^{-1}y_1^{-1}) = (A_1x_2B_1y_2C_1x_2^{-1}D_1y_2^{-1}E_1)x_1y_1x_1^{-1}y_1^{-1}$$

is still in the given condition. By the finite recursion principle, (I.5) is found. □

Theorem I.6 For any $Q \in \mathbf{P}$ nonorientable, there exists an integer $q \geq 1$ such that

$$Q \sim_{\mathrm{el}} (\prod_{i=1}^{q} x_i x_i)(= Q_q). \tag{I.6}$$

Proof Because Q is nonorientable, there is a letter x_1 in Q such that $Q = (Ax_1Bx_1C)$. By Relation 2, $D \sim_{\mathrm{el}} (AB^{-1}Cx_1x_1)$. If $(AB^{-1}C) \sim_{\mathrm{el}} (xx^{-1})$, then by Operation 0 we have $Q \sim_{\mathrm{el}} (x_1x_1)$. This is the case of $q = 1$. Otherwise, there exits an integer $k \geq 1$ such that

$$Q \sim_{\mathrm{el}} (A\prod_{i=1}^{k} x_i x_i)(= Q_q)$$

and $(A) \not\sim_{\mathrm{el}} (xx^{-1})$ is orientable. By Theorem I.5, there exists an integer $s \geq 1$ such that

$$(A) \sim_{\mathrm{el}} (\prod_{i=1}^{s} x_i y_i x_i^{-1} y_i^{-1})(= O_p).$$

Thus, by Relation 3 for s times, we have

$$Q \sim_{\mathrm{el}} (\prod_{i=1}^{2s+k} x_i x_i)(= Q_q).$$

This is $q = 2s + k \geq 1$. □

On the basis of Lemma I.8 and Theorems I.5–6, surfaces in topology are in fact the classes of polyhedra under the elementary equivalence. Surfaces O_0, O_p, $p \geq 1$, are, respectively, orientable *standard surfaces* of genus 0, p, $p \geq 1$. Surfaces Q_q, $q \geq 1$, are nonorientable standard surfaces of genus q.

Ax.I.3 Embeddings

An *embedding*(*i.e.*, *cellular embedding* in early references particularly in topology and geometry) of a graph is such a polyhedron whose skeleton is the graph.

The distinction of embeddings are the same as polyhedra. Precisely speaking, two distinct embeddings on a 2-dimensional manifold are not equivalent topologically in 1-dimensional sense.

According to Ax.I.1, all embeddings always imply to be classic.

For a graph $G = (V, E)$, Heffter-Edmonds' model of an embedding of G by rotation system at vertices, in fact, only for orientable case [Hef1] and [Edm1].

Let $\sigma = \{\sigma_v | v \in V\}$ be the rotation system on G where σ_v is the cyclic order of semi-edges at $v \in V$. Then, by the following procedure to find an embedding of G:

Procedure I.1 First, put different vertices in different position marked by a hole circle or a bold point on the plane. Draw lines for edges such that no interior point passes through a vertex and σ_v is in clockwise when v is a hole circle; in anticlockwise, otherwise.

Then, by travelling along an edge in the rule: passing through on the same side when the two ends of the edges are in same type; crossing to the other side, otherwise. Find all cycles such that each edge occurs just twice. The set of cycles is denoted by P_G.

Lemma I.9 P_G is a polyhedron.

Proof Because it is easily checked from the definition of a polyhedron. □

Lemma I.10 P_G is orientable.

Proof Because the dual is orientable, from Theorem I.4, the lemma is true. □

Theorem I.7 The dual of P_G is an orientable embedding of G.

Proof Because the support of the dual of P_G is G itself, the theorem is deduced. □

However, P_G in general is not classic except for all vertices are of same type.

Theorem I.8 For a given rotation system σ of a graph G, let $P_G(\sigma; 0)$ be the polyhedron obtained by the procedure above for all vertices of same type, then $P_G(\sigma; 0)$ is unique.

Proof From the uniqueness of classic polyhedron in this case, the theorem is done. □

On the basis of Theorem I.8, it suffices only to make all vertices with the same type, *e.g.*, in clockwise. Further, in order to extend to nonorientable case, on account of Theorem I.3, edges in a set not containing cocycle are marked for crossing one side to the other in the Heffter-Edmonds' model. The marked edges are called *twist*. This model as well as the Procedure I.1 here is called an *expansion*.

Theorem I.9 The dual of what is obtained in an expansion is a unique nonorientable embedding of G for twist edges fixed.

Proof Because one obtained in an expansion is a classic polyhedron, from the uniqueness of the dual of a polyhedron, the theorem deduced. □

Theorem I.10 All embeddings of a graph G obtained by expansions for all possible rotation system and twist edges in a subset of the cotree $\bar{T}$ of a given spanning tree T on G are distinct.

Proof As a result of Theorem I.9. □

This theorem enables us to choose a spanning tree T on a graph G for discussing all embeddings of G on surfaces.

Let T_1 and T_2 be two spanning trees of a graph G. The sets of all embeddings of G as shown in Theorem I.10 for T_1 and T_2 are, respectively, denoted by $\mathcal{E}_1$ and $\mathcal{E}_2$.

Theorem I.11 Let $\mathcal{E}_1^g$ and $\mathcal{E}_2^g$ be, respectively, the subsets of $\mathcal{E}_1$ and $\mathcal{E}_2$ on surfaces of genus g(orientable $g = p \geq 0$, or nonorientable $g = q \geq 1$). Then $\mathcal{E}_1^g = \mathcal{E}_2^g$.

Proof Because of Theorem I.10, it suffices only to discuss expansions for T_1 and T_2. Since $|\bar{T}_1| = |\bar{T}_2|$, Theorems I.8–9 implies the theorem. $\square$

For an embedding $P \in \mathcal{E}_1^g$, if $P \notin \mathcal{E}_2^g$, then there exits a twist edge e in T_2. By doing a switch with the fundamental cocircuit containing e for T_2, an embedding P' in the same distinct class with P is found. If no twit edge is in $\bar{T}_2$, then P' is the classic embedding in $\mathcal{E}_2^g$ corresponding to P. Otherwise, by the finite recursion, a classic embedding $Q \in \mathcal{E}_2^g$ in the same distinct class with P is finally found. In this way, the 1-to-1 correspondence between $\mathcal{E}_1^g$ and $\mathcal{E}_2^g$ is established.

The last two theorems form the foundation of the joint tree model shown in [Liu13–14]. Related topics are referred to [Sta1–2].

Ax.I.4 Maps

Maps as polyhedra or embeddings of its underlying graph had been being no specific meaning until 1979 when Tutte(William T., 1917–2002) clarified that a map is a particular type of permutation on a set formed as a union of quadricells[Tut1–3]. All quadricells are with similar construction that four elements have the symmetry as a straight line segment with two ends and two sides.

This idea would go back to Klein(Felix, 1849–1925) who considered a triangulation of an embedding on a surface by inserting a vertex in the interior of each face and each edge and then connecting all line segments from a vertex in the interior of a face to all vertices on the boundary of the face. It is seen that each edge is adjacent to four triangles called *flags* as a quadricell. So, such a pattern of map used in this course can be named as Klein–Tutte's model. Related topics are referred to [Vin1–2].

Now, we have seen that a surfaces is determined by an elementary class of polyhedra, an embedding is by a distinct class of polyhedra and a map is by an isomorphic class of embeddings. The distinction of embeddings is based on edges labelled by letters, or numbers. This is also

a kind of asymmetrization. But edges on a map are without labelling. Isomorphic maps are combinatorially considered with symmetry. So, a map is an isomorphic class of embeddings of its underlying graph.

Let $G = (V, E)$ be a graph. As shown in Sect. 1.1, $V = \mathrm{Par}(X)$ and $E = \{Bx|x \in X\}$ where $\mathrm{Par}(X)$ is a partition on $B(X) = \cup_{x\in X} Bx$, $Bx = \{x(0), x(1)\}$ for a set X. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if, and only if, there exists a bijection ι: $X_1 \to X_2$ such that the diagrams

$$\begin{tikzcd} X_1 \arrow[r, "\iota"] \arrow[d, "\sigma_1"'] & X_2 \arrow[d, "\sigma_2"] \\ X_1 \arrow[r, "\iota"'] & X_2 \end{tikzcd} \tag{I.7}$$

for $\sigma_i = B_i, \mathrm{Par}_i$, $i = 1, 2$, are commutative. Let $\mathrm{Aut}(G)$ be the automorphism group of G.

On the other hand, a *semi-arc isomorphism* between two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined to be such a bijection τ: $B_1(X_1) \to B_2(X_2)$ that

$$\begin{tikzcd} B_1(X_1) \arrow[r, "\tau"] \arrow[d, "\sigma_1"'] & B_2(X_2) \arrow[d, "\sigma_2"] \\ B_1(X_1) \arrow[r, "\tau"'] & B_2(X_2) \end{tikzcd} \tag{I.8}$$

for $\sigma_i = B_i, \mathrm{Par}_i$, $i = 1, 2$, are commutative. Let $\mathrm{Aut}_{1/2}(G)$ be the semi-arc automorphism group of G.

Theorem I.12 If $\mathrm{Aut}(G)$ and $\mathrm{Aut}_{1/2}(G)$ are, resp., the automorphism and semi-arc automorphism groups of graph G, then

$$\mathrm{Aut}_{1/2}(G) = \mathrm{Aut}(G) \times S_2^l \tag{I.9}$$

where l is the number of self-loops on G and S_2 is the symmetric group of degree 2.

Proof Because each automorphism of G just induces two semi-arc isomorphisms of G for a self-loop, the theorem is true. □

For map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, its automorphisms are discussed with asymmetrization in Chapter VIII. Let $\mathcal{M}(G)$ be the set of all nonisomorphic maps with underlying graph G.

Lemma I.11 For an automorphism ζ on map $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$, we have exhaustively $\zeta|_{B(X)} \in \mathrm{Aut}_{1/2}(G)$ and $\zeta\alpha|_{B(X)} \in \mathrm{Aut}_{1/2}(G)$ where $G = G(M)$, the underlying graph of M, and $B(X) = X + \beta X$.

Proof Because $\mathcal{X}_{\alpha,\beta}(X) = (X + \beta X) + (\alpha X + \alpha\beta X) = B(X) + \alpha B(X)$, by Conjugate Axiom each $\zeta \in \mathrm{Aut}(M)$ has exhaustively two possibilities: $\zeta|_{B(X)} \in \mathrm{Aut}_{1/2}(G)$ and $\zeta\alpha|_{B(X)} \in \mathrm{Aut}_{1/2}(G)$. $\square$

Theorem I.13 Let $\mathcal{E}_g(G)$ be the set of all embeddings of a graph G on a surface of genus g(orientable or nonorientable), then the number of nonisomorphic maps in $\mathcal{E}_g(G)$ is

$$m_g(G) = \frac{1}{2 \times \mathrm{aut}_{1/2}(G)} \sum_{\tau \in \mathrm{Aut}_{1/2}(G)} |\Phi(\tau)| \tag{I.10}$$

where $\Phi(\tau) = \{M \in \mathcal{E}_g(G) | \tau(M) = M \text{ or } \tau\alpha(M) = M\}$.

Proof Suppose $X_1, X_2, \cdots, X_m$ are all the equivalent classes of $X = \mathcal{E}_g(G)$ under the group $\mathrm{Aut}_{1/2}(G) \times \langle\alpha\rangle$, then $m = m_g(G)$. Let

$$S(x) = \{\tau \in \mathrm{Aut}_{1/2}(G) \times \langle\alpha\rangle |\ \tau(x) = x\}$$

be the stabilizer at x, a subgroup of $\mathrm{Aut}_{1/2}(G)\times\langle\alpha\rangle$. Because $|\mathrm{Aut}_{1/2}(G)\times\langle\alpha\rangle| = |S(x_i)||X_i|$, $x_i \in X_i$, $i = 1, 2, \cdots, m$, we have

$$m|\mathrm{Aut}_{1/2}(G) \times \langle\alpha\rangle| = \sum_{i=1}^{m} |S(x_i)||X_i|. \tag{1}$$

By observing $|S(x_i)|$ independent of the choice of x_i in the class X_i,

the right hand side of (1) is

$$\begin{aligned}\sum_{x\in X}|S(x)| &= \sum_{x\in X}\sum_{\tau\in S(x)}1\\ &= \sum_{\tau\in \mathrm{Aut}_{1/2}(G)\times\langle\alpha\rangle}\ \sum_{x=\tau(x)}1\\ &= \sum_{\tau\in \mathrm{Aut}_{1/2}(G)\times\langle\alpha\rangle}|\Phi(\tau)|.\end{aligned} \tag{2}$$

From (1) and (2), the theorem can be soon derived. □

The theorem above shows how to find nonisomorphic super maps of a graph when the automorphism group of the graph is known.

Theorem I.14 For a graph G, let $\mathcal{R}_g(G)$ and $\mathcal{E}_g(G)$ be, respectively, the sets of all nonisomorphic rooted super maps and all distinct embeddings of G with size $\epsilon(G)$ on a surface of genus g(orientable or nonorientable). Then,

$$|\mathcal{R}_g(G)| = \frac{2\epsilon(G)}{\mathrm{aut}_{1/2}(G)}|\mathcal{E}_g(G)|. \tag{I.11}$$

Proof Let $\mathcal{M}_g(G)$ be the set of all nonisomorphic super maps of G. By (11.3),

$$|\mathcal{R}_g(G)| = \frac{4\epsilon(G)}{2\times \mathrm{aut}_{1/2}(G)}\sum_{M\in\mathcal{M}_g(G)}\frac{2\times \mathrm{aut}_{1/2}(G)}{\mathrm{aut}(M)}.$$

By considering that

$$\begin{aligned}2\times \mathrm{aut}_{1/2}(G) &= |\mathrm{Aut}_{1/2}(G)\times\langle\alpha\rangle|\\ &= |(\mathrm{Aut}_{1/2}(G)\times\langle\alpha\rangle)|_M|\times|\mathrm{Aut}_{1/2}(G)\times\langle\alpha\rangle(M)|\end{aligned}$$

and $(\mathrm{Aut}_{1/2}(G)\times\langle\alpha\rangle)|_M = \mathrm{Aut}(M)$, we have

$$\begin{aligned}|\mathcal{R}_g(G)| &= \frac{4\epsilon(G)}{|\mathrm{Aut}_{1/2}(G)\times\langle\alpha\rangle|}\sum_{M\in\mathcal{M}_g(G)}|\mathrm{Aut}_{1/2}(G)\times\langle\alpha\rangle(M)|\\ &= \frac{2\epsilon(G)}{\mathrm{aut}_{1/2}(G)}|\mathcal{E}_g(G)|.\end{aligned}$$

This is (I.11). □

This theorem enables us to determine all the super rooted maps of a graph when the automorphism group of the graph is known. However, the problem of finding an automorphism of a graph is much more difficult than that of finding an automorphism of a map on the basis of Chapter VIII in general. For asymmetric graphs the two theorems above provide results much simpler. More results are referred to [MLW1].

Appendix II

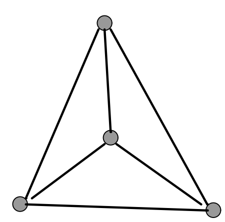
Table of Genus Polynomials for Embeddings and Maps of Small Size

For a graph G, let $p_G(x)$, $\mu_G(x)$ and $\mu^r_G(x)$ be, respectively, the orientable genus distributions of embeddings, super maps and rooted super maps of G, or called *orientable genus polynomials*. Similarly, let $q_G(x^{-1})$, $\nu_G(x^{-1})$ and $\nu^r_G(x^{-1})$ be, respectively, the nonorientable genus distributions of embeddings, super maps and rooted super maps of G, or called *nonorientable genus polynomials*.

Ax.II.1 Triconnected cubic graphs

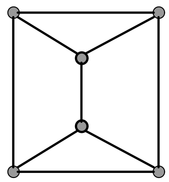
First, list all nonisomorphic 3-connected cubic graphs from size 6 through 15.

Size 6

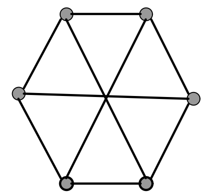


$C_{6,1}$

Size 9

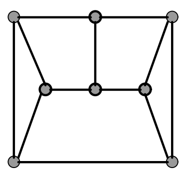


$C_{9,1}$

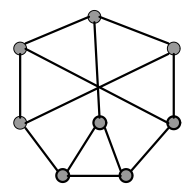


$C_{9,2}$

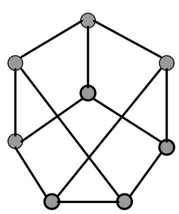
Size 12



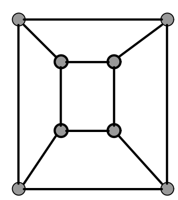
$C_{12,1}$



$C_{12,2}$

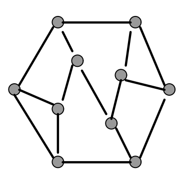


$C_{12.3}$

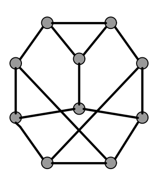


$C_{12,4}$

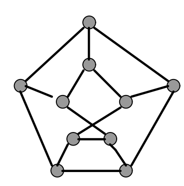
Size 15



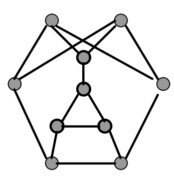
$C_{15,1}$



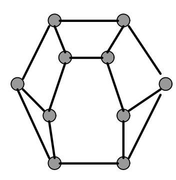
$C_{15,2}$



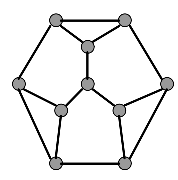
$C_{15,3}$



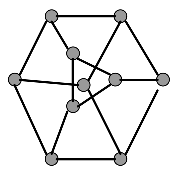
$C_{15,4}$



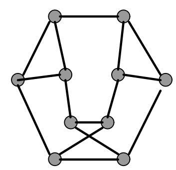
$C_{15,5}$



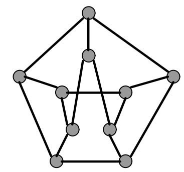
$C_{15,6}$



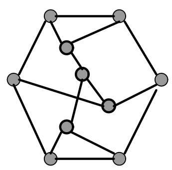
$C_{15,7}$



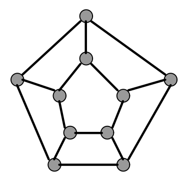
$C_{15,8}$



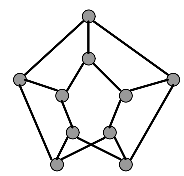
$C_{15,9}$



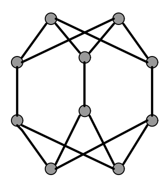
$C_{15,10}$



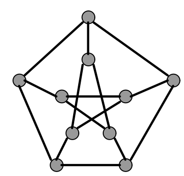
$C_{15,11}$



$C_{15,12}$



$C_{15,13}$



$C_{15,14}$

In what follows, the orientable and nonorientable genus polynomials of embeddings, super maps and rooted super maps of 3-connected cubic graphs shown above are provided.

Case of size 6:

Orientable

$$p_{C_{6,1}}(x) = 2 + 14x,$$
$$\mu_{C_{6,1}}(x) = 1 + 2x,$$
$$\mu^r_{C_{6,1}}(x) = 1 + 7x.$$

Nonorientable

$$q_{C_{6,1}}(x^{-1}) = 14x + 42x^2 + 56x^3,$$
$$\nu_{C_{6,1}}(x^{-1}) = 2x + 3x^2 + 3x^3,$$
$$\nu^r_{C_{6,1}}(x^{-1}) = 7x + 21x^2 + 28x^3.$$

Case of size 9:

Orientable

$$\begin{aligned}
p_{C_{9,1}}(x) &= 2 + 38x + 24x^2,\\
\mu_{C_{9,1}}(x) &= 1 + 5x + 2x^2,\\
\mu^r_{C_{9,1}}(x) &= 3 + 57x + 36x^2;
\end{aligned}$$

$$\begin{aligned}
p_{C_{9,2}}(x) &= 40x + 24x^2,\\
\mu_{C_{9,2}}(x) &= 2x + x^2,\\
\mu^r_{C_{9,2}}(x) &= 10x + 6x^2.
\end{aligned}$$

Nonorientable

$$\begin{aligned}
q_{C_{9,1}}(x^{-1}) &= 22x + 122x^2 + 424x^3 + 392x^4,\\
\nu_{C_{9,1}}(x^{-1}) &= 3x + 12x^2 + 28x^3 + 23x^4,\\
\nu^r_{C_{9,1}}(x^{-1}) &= 33x + 183x^2 + 636x^3 + 588x^4;
\end{aligned}$$

$$\begin{aligned}
q_{C_{9,2}}(x^{-1}) &= 12x + 108x^2 + 432x^3 + 408x^4,\\
\nu_{C_{9,2}}(x^{-1}) &= x + 2x^2 + 6x^3 + 6x^4;\\
\nu^r_{C_{9,2}}(x^{-1}) &= 3x + 27x^2 + 108x^3 + 102x^4.
\end{aligned}$$

Case of size 12:

Orientable

$$\begin{aligned}
p_{C_{12,1}}(x) &= 2 + 70x + 184x^2,\\
\mu_{C_{12,1}}(x) &= 1 + 15x + 28x^2,\\
\mu^r_{C_{12,1}}(x) &= 12 + 420x + 1104x^2;
\end{aligned}$$

$$p_{C_{12,2}}(x) = 64x + 192x^2,$$
$$\mu_{C_{12,2}}(x) = 4x + 12x^2,$$
$$\mu^r_{C_{12,2}}(x) = 128x + 384x^2;$$

$$p_{C_{12,3}}(x) = 56x + 200x^2,$$
$$\mu_{C_{12,3}}(x) = 5x + 13x^2,$$
$$\mu^r_{C_{12,3}}(x) = 84x + 300x^2;$$

$$p_{C_{12,4}}(x) = 2 + 54x + 200x^2,$$
$$\mu_{C_{12,4}}(x) = 1 + 5x + 8x^2,$$
$$\mu^r_{C_{12,4}}(x) = 1 + 27x + 100x^2.$$

Nonorientable

$$q_{C_{12,1}}(x^{-1}) = 30x + 242x^2 + 1448x^3 + 3272x^4 + 2944x^5,$$
$$\nu_{C_{12,1}}(x^{-1}) = 7x + 44x^2 + 217x^3 + 452x^4 + 38x^5,$$
$$\nu^r_{C_{12,1}}(x^{-1}) = 180x + 1452x^2 + 8688x^3 + 19632x^4 + 17664x^5;$$

$$q_{C_{12,2}}(x^{-1}) = 12x + 180x^2 + 1360x^3 + 3312x^4 + 3072x^5,$$
$$\nu_{C_{12,2}}(x^{-1}) = x + 9x^2 + 64x^3 + 149x^4 + 137x^5,$$
$$\nu^r_{C_{12,2}}(x^{-1}) = 24x + 360x^2 + 2720x^3 + 6624x^4 + 6144x^5;$$

$$q_{C_{12,3}}(x^{-1}) = 10x + 158x^2 + 1272x^3 + 3296x^4 + 3200x^5,$$
$$\nu_{C_{12,3}}(x^{-1}) = 2x + 11x^2 + 57x^3 + 133x^4 + 118x^5;$$
$$\nu^r_{C_{12,3}}(x^{-1}) = 15x + 237x^2 + 1908x^3 + 2944x^4 + 4800x^5;$$

$$q_{C_{12,4}}(x^{-1}) = 24x + 192x^2 + 1288x^3 + 3264x^4 + 3168x^5,$$
$$\nu_{C_{12,4}}(x^{-1}) = x + 7x^2 + 24x^3 + 58x^4 + 40x^5;$$
$$\nu^r_{C_{12,4}}(x^{-1}) = 12x + 96x^2 + 644x^3 + 1632x^4 + 1584x^5.$$

Case of size 15:

Orientable

$$p_{C_{15,1}}(x) = 2 + 102x + 664x^2 + 256x^3,$$
$$\mu_{C_{15,1}}(x) = 1 + 27x + 176x^2 + 68x^3,$$
$$\mu^r_{C_{15,1}}(x) = 30 + 1530x + 9960x^2 + 3840x^3;$$

$$p_{C_{15,2}}(x) = 72x + 664x^2 + 288x^3,$$
$$\mu_{C_{15,2}}(x) = 20x + 180x^2 + 672x^3,$$
$$\mu^r_{C_{15,2}}(x) = 1080x + 9960x^2 + 4320x^3;$$

$$p_{C_{15,3}}(x) = 56x + 648x^2 + 320x^3,$$
$$\mu_{C_{15,3}}(x) = 12x + 96x^2 + 44x^3,$$
$$\mu^r_{C_{15,3}}(x) = 420x + 4860x^2 + 2400x^3;$$

$$p_{C_{15,4}}(x) = 80x + 688x^2 + 256x^3,$$
$$\mu_{C_{15,4}}(x) = 11x + 93x^2 + 32x^3,$$
$$\mu^r_{C_{15,4}}(x) = 600x + 5160x^2 + 1920x^3;$$

$$p_{C_{15,5}}(x) = 2 + 118x + 648x^2 + 256x^3,$$
$$\mu_{C_{15,5}}(x) = 1 + 27x + 88x^2 + 36x^3,$$
$$\mu^r_{C_{15,5}}(x) = 15 + 885x + 4860x^2 + 1920x^3;$$

$$p_{C_{15,6}}(x) = 2 + 110x + 688x^2 + 224x^3,$$
$$\mu_{C_{15,6}}(x) = 1 + 14x + 69x^2 + 20x^3,$$
$$\mu^r_{C_{15,6}}(x) = 10 + 550x + 3440x^2 + 1120x^3;$$

$$p_{C_{15,7}}(x) = 2 + 78x + 656x^2 + 288x^3,$$
$$\mu_{C_{15,7}}(x) = 1 + 14x + 81x^2 + 24x^3,$$
$$\mu^r_{C_{15,7}}(x) = 10 + 390x + 3280x^2 + 1440x^3;$$

$$p_{C_{15,8}}(x) = 96x + 672x^2 + 256x^3,$$
$$\mu_{C_{15,8}}(x) = 9x + 49x^2 + 18x^3,$$
$$\mu^r_{C_{15,8}}(x) = 360x + 2520x^2 + 960x^3;$$

$$\begin{aligned}
p_{C_{15,9}}(x) &= 48x + 656x^2 + 320x^3,\\
\mu_{C_{15,9}}(x) &= 8x + 59x^2 + 25x^3,\\
\mu^r_{C_{15,9}}(x) &= 180x + 2460x^2 + 1200x^3;
\end{aligned}$$

$$\begin{aligned}
p_{C_{15,10}}(x) &= 88x + 648x^2 + 288x^3,\\
\mu_{C_{15,10}}(x) &= 5x + 31x^2 + 16x^3,\\
\mu^r_{C_{15,10}}(x) &= 220x + 1620x^2 + 720x^3;
\end{aligned}$$

$$\begin{aligned}
p_{C_{15,11}}(x) &= 2 + 70x + 632x^2 + 320x^3,\\
\mu_{C_{15,11}}(x) &= 1 + 5x + 28x^2 + 10x^3,\\
\mu^r_{C_{15,11}}(x) &= 3 + 105x + 948x^2 + 480x^3;
\end{aligned}$$

$$\begin{aligned}
p_{C_{15,12}}(x) &= 72x + 632x^2 + 320x^3,\\
\mu_{C_{15,12}}(x) &= 6x + 24x^2 + 14x^3,\\
\mu^r_{C_{15,12}}(x) &= 108x + 948x^2 + 480x^3;
\end{aligned}$$

$$\begin{aligned}
p_{C_{15,13}}(x) &= 48x + 720x^2 + 256x^3,\\
\mu_{C_{15,13}}(x) &= 2x + 15x^2 + 6x^3,\\
\mu^r_{C_{15,13}}(x) &= 30x + 450x^2 + 160x^3;
\end{aligned}$$

$$\begin{aligned}
p_{C_{15,14}}(x) &= 40x + 664x^2 + 320x^3,\\
\mu_{C_{15,14}}(x) &= x + 7x^2 + 2x^3,\\
\mu^r_{C_{15,14}}(x) &= 10x + 166x^2 + 80x^3.
\end{aligned}$$

Nonorientable

$$\begin{aligned}
q_{C_{15,1}}(x^{-1}) = 38x + 394x^2 + 3336x^3 + 12744x^4 + 27008x^5\\
+ 20992x^6,
\end{aligned}$$

$$\begin{aligned}
\nu_{C_{15,1}}(x^{-1}) &= 10x + 104x^2 + 838x^3 + 3220x^4 + 6768x^5 + 5300x^6;\\
\nu^r_{C_{15,1}}(x^{-1}) &= 570x + 5910x^2 + 50040x^3 + 191160x^4 + 405120x^5\\
&\quad + 314880x^6;
\end{aligned}$$

$$q_{C_{15,2}}(x^{-1}) = 10x + 214x^2 + 2576x^3 + 11664x^4 + 27424x^5 + 22624x^6,$$
$$\nu_{C_{15,2}}(x^{-1}) = 4x + 60x^2 + 676x^3 + 2988x^4 + 6952x^5 + 5688x^6;$$
$$\nu^r_{C_{15,2}}(x^{-1}) = 150x + 3210x^2 + 38640x^3 + 174960x^4 + 411360x^5 + 339360x^6;$$

$$q_{C_{15,3}}(x^{-1}) = 6x + 158x^2 + 2188x^3 + 10912x^4 + 27504x^5 + 23744x^6,$$
$$\nu_{C_{15,3}}(x^{-1}) = 2x + 27x^2 + 313x^3 + 1466x^4 + 3572x^5 + 3044x^6;$$
$$\nu^r_{C_{15,3}}(x^{-1}) = 45x + 1185x^2 + 16410x^3 + 81840x^4 + 206280x^5 + 178080x^6;$$

$$q_{C_{15,4}}(x^{-1}) = 12x + 244x^2 + 2816x^3 + 12224x^4 + 27456x^5 + 21760x^6,$$
$$\nu_{C_{15,4}}(x^{-1}) = 2x + 33x^2 + 368x^3 + 1565x^4 + 3480x^5 + 2736x^6;$$
$$\nu^r_{C_{15,4}}(x^{-1}) = 90x + 1830x^2 + 21120x^3 + 91680x^4 + 205920x^5 + 163200x^6;$$

$$q_{C_{15,5}}(x^{-1}) = 38x + 410x^2 + 3496x^3 + 12952x^4 + 26880x^5 + 20736x^6,$$
$$\nu_{C_{15,5}}(x^{-1}) = 8x + 76x^2 + 524x^3 + 1768x^4 + 3460x^5 + 2652x^6;$$
$$\nu^r_{C_{15,5}}(x^{-1}) = 385x + 3075x^2 + 26220x^3 + 97140x^4 + 201600x^5 + 155520x^6;$$

$$q_{C_{15,6}}(x^{-1}) = 38x + 402x^2 + 3448x^3 + 13040x^4 + 27072x^5 + 20512x^6,$$
$$\nu_{C_{15,6}}(x^{-1}) = 6x + 44x^2 + 319x^3 + 1157x^4 + 2354x^5 + 1744x^6;$$
$$\nu^r_{C_{15,6}}(x^{-1}) = 190x + 2010x^2 + 17240x^3 + 65200x^4 + 135360x^5 + 102560x^6;$$

$$\begin{aligned} q_{C_{15,7}}(x^{-1}) &= 32x + 312x^2 + 2800x^3 + 11800x^4 + 27200x^5 \\ &\quad + 22368x^6, \\ \nu_{C_{15,7}}(x^{-1}) &= 5x + 35x^2 + 267x^3 + 1077x^4 + 2358x^5 + 1866x^6; \\ \nu^r_{C_{15,7}}(x^{-1}) &= 160x + 1560x^2 + 14000x^3 + 59000x^4 + 136000x^5 \\ &\quad + 111840x^6; \end{aligned}$$

$$\begin{aligned} q_{C_{15,8}}(x^{-1}) &= 12x + 260x^2 + 2976x^3 + 12432x^4 + 27328x^5 \\ &\quad + 21504x^6, \\ \nu_{C_{15,8}}(x^{-1}) &= 2x + 21x^2 + 207x^3 + 828x^4 + 1772x^5 + 1382x^6; \\ \nu^r_{C_{15,8}}(x^{-1}) &= 45x + 975x^2 + 11160x^3 + 26620x^4 + 192480x^5 \\ &\quad + 80640x^6; \end{aligned}$$

$$\begin{aligned} q_{C_{15,9}}(x^{-1}) &= 4x + 132x^2 + 2049x^3 + 10720x^4 + 27616x^5 \\ &\quad + 24000x^6, \\ \nu_{C_{15,9}}(x^{-1}) &= x + 16x^2 + 152x^3 + 753x^4 + 1811x^5 + 1559x^6; \\ \nu^r_{C_{15,9}}(x^{-1}) &= 15x + 495x^2 + 7650x^3 + 40200x^4 + 103560x^5 \\ &\quad + 90000x^6; \end{aligned}$$

$$\begin{aligned} q_{C_{15,10}}(x^{-1}) &= 12x + 252x^2 + 2864x^3 + 12136x^4 + 27264x^5 \\ &\quad + 21984x^6, \\ \nu_{C_{15,10}}(x^{-1}) &= x + 11x^2 + 124x^3 + 517x^4 + 1154x^5 + 941x^6; \\ \nu^r_{C_{15,10}}(x^{-1}) &= 30x + 630x^2 + 7160x^3 + 30340x^4 + 681560x^5 \\ &\quad + 54960x^6; \end{aligned}$$

$$\begin{aligned} q_{C_{15,11}}(x^{-1}) &= 30x + 282x^2 + 2560x^3 + 11240x^4 + 27168x^5 \\ &\quad + 23232x^6, \\ \nu_{C_{15,11}}(x^{-1}) &= 2x + 17x^2 + 92x^3 + 351x^4 + 754x^5 + 624x^6; \\ \nu^r_{C_{15,11}}(x^{-1}) &= 45x + 423x^2 + 3840x^3 + 16860x^4 + 40752x^5 \\ &\quad + 34848x^6; \end{aligned}$$

$$\begin{aligned} q_{C_{15,12}}(x^{-1}) &= 12x + 220x^2 + 2480x^3 + 11240x^4 + 27264x^5 \\ &\quad + 23296x^6, \\ \nu_{C_{15,12}}(x^{-1}) &= 2x + 11x^2 + 90x^3 + 343x^4 + 756x^5 + 638x^6; \\ \nu^r_{C_{15,12}}(x^{-1}) &= 18x + 330x^2 + 3720x^3 + 16860x^4 + 40896x^5 \\ &\quad + 34944x^6; \end{aligned}$$

$$\begin{aligned} q_{C_{15,13}}(x^{-1}) &= 120x^2 + 2232x^3 + 11568x^4 + 27936x^5 \\ &\quad + 22656x^6, \\ \nu_{C_{15,13}}(x^{-1}) &= 4x^2 + 28x^3 + 144x^4 + 307x^5 + 259x^6; \\ \nu^r_{C_{15,13}}(x^{-1}) &= 75x^2 + 1395x^3 + 7230x^4 + 17460x^5 \\ &\quad + 14160x^6; \end{aligned}$$

$$\begin{aligned} q_{C_{15,14}}(x^{-1}) &= 4x + 120x^2 + 1900x^3 + 10440x^4 + 27664x^5 \\ &\quad + 24384x^6, \\ \nu_{C_{15,14}}(x^{-1}) &= x + 2x^2 + 16x^3 + 62x^4 + 142x^5 + 111x^6; \\ \nu^r_{C_{15,14}}(x^{-1}) &= x + 30x^2 + 474x^3 + 2610x^4 + 6916x^5 \\ &\quad + 6039x^6. \end{aligned}$$

Ax.II.2 Bouquets

Let B_m be the bouquet of size m, $m \geq 1$.

Case of $m = 1$:

Orientable

$$\begin{aligned} p_{B_1}(x) &= 1, \\ \mu_{B_1}(x) &= 1, \\ \mu^r_{B_1}(x) &= 1. \end{aligned}$$

Nonorientable

$$q_{B_1}(x^{-1}) = x,$$
$$\nu_{B_1}(x^{-1}) = x,$$
$$\nu^r_{B_1}(x^{-1}) = x.$$

Case of $m = 2$:

Orientable

$$p_{B_2}(x) = 4 + 2x,$$
$$\mu_{B_2}(x) = 1 + x,$$
$$\mu^r_{B_2}(x) = 2 + x.$$

Nonorientable

$$q_{B_2}(x^{-1}) = 10x + 8x^2,$$
$$\nu_{B_2}(x^{-1}) = 2x + 2x^2,$$
$$\nu^r_{B_2}(x^{-1}) = 5x + 4x^2.$$

Case of $m = 3$:

Orientable

$$p_{B_3}(x) = 40 + 80x,$$
$$\mu_{B_3}(x) = 2 + 3x,$$
$$\mu^r_{B_3}(x) = 5 + 10x.$$

Nonorientable

$$q_{B_3}(x^{-1}) = 176x + 336x^2 + 328x^3,$$

$$\nu_{B_3}(x^{-1}) = 5x + 8x^2 + 8x^3,$$
$$\nu^r_{B_3}(x^{-1}) = 22x + 42x^2 + 41x^3.$$

Case of $m = 4$**:**

Orientable

$$p_{B_4}(x) = 672 + 3360x + 1008x^2,$$
$$\mu_{B_4}(x) = 3 + 10x + 4x^2,$$
$$\mu^r_{B_4}(x) = 14 + 70x + 21x^2.$$

Nonorientable

$$q_{B_4}(x^{-1}) = 4464x + 14592x^2 + 33120x^3 + 23424x^4,$$
$$\nu_{B_4}(x^{-1}) = 12x + 33x^2 + 64x^3 + 47x^4,$$
$$\nu^r_{B_4}(x^{-1}) = 93x + 304x^2 + 690x^3 + 488x^4.$$

Case of $m = 5$**:**

Orientable

$$p_{B_5}(x) = 16128 + 161280x + 185472x^2,$$
$$\mu_{B_5}(x) = 6 + 35x + 38x^2,$$
$$\mu^r_{B_5}(x) = 42 + 420x + 483x^2.$$

Nonorientable

$$q_{B_5}(x^{-1}) = 148224x + 718080x^2 + 2745600x^3 + 4477440x^4 + 3159936x^5,$$
$$\nu_{B_5}(x^{-1}) = 33x + 131x^2 + 442x^3 + 686x^4 + 473x^5,$$
$$\nu^r_{B_5}(x^{-1}) = 386x + 1870x^2 + 7150x^3 + 11660x^4 + 8229x^5.$$

Case of $m = 6$:

Orientable

$$\begin{aligned}
p_{B_6}(x) &= 506880 + 8870400x + 24837120x^2 + 5702400x^3,\\
\mu_{B_6}(x) &= 12 + 132x + 328x^2 + 82x^3,\\
\mu^r_{B_6}(x) &= 132 + 2310x + 6468x^2 + 1485x^3.
\end{aligned}$$

Ax.II.3 Wheels

Let W_n be the wheel of order n, $n \geq 4$, *i.e.*, all vertices are of valency (or degree) 3 but one and all 3-valent vertices form a circuit.

Case of $n = 4$:

Orientable

$$\begin{aligned}
p_{W_4}(x) &= 2 + 14x,\\
\mu_{W_4}(x) &= 1 + 2x,\\
\mu^r_{W_4}(x) &= 1 + 7x.
\end{aligned}$$

Nonorientable

$$\begin{aligned}
q_{W_4}(x^{-1}) &= 14x + 42x^2 + 56x^3,\\
\nu_{W_4}(x^{-1}) &= 2x + 3x^2 + 3x^3,\\
\nu^r_{W_4}(x^{-1}) &= 7x + 21x^2 + 28x^3.
\end{aligned}$$

Case of $n = 5$:

Orientable

$$\begin{aligned}
p_{W_5}(x) &= 2 + 58x + 36x^2,\\
\mu_{W_5}(x) &= 1 + 8x + 4x^2,\\
\mu^r_{W_5}(x) &= 4 + 116x + 72x^2.
\end{aligned}$$

Nonorientable

$$q_{W_5}(x^{-1}) = 28x + 176x^2 + 640x^3 + 596x^4,$$
$$\nu_{W_5}(x^{-1}) = 4x + 18x^2 + 52x^3 + 48x^4,$$
$$\nu^r_{W_5}(x^{-1}) = 56x + 352x^2 + 1280x^3 + 1192x^4.$$

Case of $n = 6$:

Orientable

$$p_{W_6}(x) = 2 + 190x + 576x^2,$$
$$\mu_{W_6}(x) = 1 + 14x + 41x^2,$$
$$\mu^r_{W_6}(x) = 4 + 380x + 1152x^2.$$

Nonorientable

$$q_{W_6}(x^{-1}) = 52x + 580x^2 + 4080x^3 + 9880x^4 + 9216x^5,$$
$$\nu_{W_6}(x^{-1}) = 6x + 38x^2 + 227x^3 + 539x^4 + 494x^5,$$
$$\nu^r_{W_6}(x^{-1}) = 104x + 1160x^2 + 8160x^3 + 19760x^4 + 18432x^5.$$

Case of $n = 7$:

Orientable

$$p_{W_7}(x) = 2 + 550x + 4968x^2 + 2160x^3,$$
$$\mu_{W_7}(x) = 1 + 34x + 240x^2 + 106x^3,$$
$$\mu^r_{W_7}(x) = 4 + 1100x + 9936x^2 + 4320x^3.$$

Nonorientable

$$q_{W_7}(x^{-1}) = 94x + 1680x^2 + 19482x^3 + 87536x^4 + 205496x^5 + 169552x^6,$$

$$\nu_{W_7}(x^{-1}) = 8x + 89x^2 + 878x^3 + 3829x^4 + 8788x^5 + 7241x^6,$$
$$\nu^r_{W_7}(x^{-1}) = 188x + 3360x^2 + 38964x^3 + 175072x^4 + 410992x^5 + 339104x^6.$$

Case of order $n = 8$**:**

Orientable

$$p_{W_8}(x) = 2 + 1484x + 31178x^2 + 59496x^3,$$
$$\mu_{W_8}(x) = 1 + 63x + 1176x^2 + 2246x^3,$$
$$\mu^r_{W_8}(x) = 4 + 2968x + 62356x^2 + 118992x^3.$$

Ax.II.4 Link bundles

Let L_m be the link bundle of size m, $m \geq 3$. A *link bundle* is a graph of order 2 without loop.

Case of size $m = 3$**:**

Orientable

$$p_{L_3}(x) = 2 + 2x,$$
$$\mu_{L_3}(x) = 1 + x,$$
$$\mu^r_{L_3}(x) = 1 + 1x.$$

Nonorientable

$$q_{L_3}(x^{-1}) = 6x + 6x^2,$$
$$\nu_{L_3}(x^{-1}) = 1x + 2x^2,$$
$$\nu^r_{L_3}(x^{-1}) = 3x + 3x^2.$$

Case of size $m = 4$**:**

Orientable

$$
\begin{aligned}
p_{L_4}(x) &= 6 + 30x,\\
\mu_{L_4}(x) &= 1 + 2x,\\
\mu^r_{L_4}(x) &= 1 + 5x.
\end{aligned}
$$

Nonorientable

$$
\begin{aligned}
q_{L_4}(x^{-1}) &= 36x + 96x^2 + 120x^3,\\
\nu_{L_4}(x^{-1}) &= 2x + 4x^2 + 3x^3,\\
\nu^r_{L_4}(x^{-1}) &= 6x + 16x^2 + 20x^3.
\end{aligned}
$$

Case of size $m = 5$**:**

Orientable

$$
\begin{aligned}
p_{L_5}(x) &= 24 + 360x + 192x^2,\\
\mu_{L_5}(x) &= 1 + 3x + 3x^2,\\
\mu^r_{L_5}(x) &= 1 + 15x + 8x^2.
\end{aligned}
$$

Nonorientable

$$
\begin{aligned}
q_{L_5}(x^{-1}) &= 240x + 1200x^2 + 3840x^3 + 3360x^4,\\
\nu_{L_5}(x^{-1}) &= 2x + 7x^2 + 14x^3 + 14x^4,\\
\nu^r_{L_5}(x^{-1}) &= 10x + 50x^2 + 160x^3 + 140x^4.
\end{aligned}
$$

Case of size $m = 6$**:**

Orientable

$$
\begin{aligned}
p_{L_6}(x) &= 120 + 4200x + 10080x^2,\\
\mu_{L_6}(x) &= 1 + 6x + 10x^2,\\
\mu^r_{L_6}(x) &= 1 + 35x + 84x^2.
\end{aligned}
$$

Nonorientable

$$q_{L_6}(x^{-1}) = 1800x + 14400x^2 + 84120x^3 + 184320x^4 + 161760x^5,$$
$$\nu_{L_6}(x^{-1}) = 3x + 14x^2 + 48x^3 + 96x^4 + 72x^5,$$
$$\nu^r_{L_6}(x^{-1}) = 15x + 120x^2 + 701x^3 + 1536x^4 + 1348x^5.$$

Case of size $m = 7$**:**

Orientable

$$p_{L_7}(x) = 720 + 50400x + 337680x^2 + 129600x^3,$$
$$\mu_{L_7}(x) = 1 + 8x + 31x^2 + 16x^3,$$
$$\mu^r_{L_7}(x) = 1 + 70x + 469x^2 + 180x^3.$$

Ax.II.5 Complete bipartite graphs

Let $K_{m,n}$ be the complete bipartite graph of order $m+n$, $m, n \geq 3$.

Case of order $m + n = 6$**:**

Orientable

$$p_{K_{3,3}}(x) = 40x + 24x^2,$$
$$\mu_{K_{3,3}}(x) = 2x + x^2,$$
$$\mu^r_{K_{3,3}}(x) = 10x + 6x^2.$$

Nonorientable

$$q_{K_{3,3}}(x^{-1}) = 12x + 108x^2 + 432x^3 + 408x^4,$$
$$\nu_{K_{3,3}}(x^{-1}) = x + 2x^2 + 6x^3 + 6x^4,$$
$$\nu^r_{K_{3,3}}(x^{-1}) = 3x + 27x^2 + 108x^3 + 102x^4.$$

Case of order $m+n=7$:

Orientable

$$p_{K_{3,4}}(x) = 156x + 2244x^2 + 1056x^3,$$
$$\mu_{K_{3,4}}(x) = 3x + 16x^2 + 10x^3,$$
$$\mu^r_{K_{3,4}}(x) = 26x + 374x^2 + 176x^3.$$

Nonorientable

$$q_{K_{3,4}}(x^{-1}) = 12x + 432x^2 + 6852x^3 + 36288x^4 + 93360x^5 + 80784x^6,$$
$$\nu_{K_{3,4}}(x^{-1}) = x + 4x^2 + 33x^3 + 156x^4 + 358x^5 + 317x^6,$$
$$\nu^r_{K_{3,4}}(x^{-1}) = 2x + 72x^2 + 1142x^3 + 6048x^4 + 15560x^5 + +13464x^6.$$

Case of order $m+n=8$:

Orientable

$$p_{K_{4,4}}(x) = 108x + 24984x^2 + 565020x^3 + 1089504x^4,$$
$$\mu_{K_{4,4}}(x) = 2x + 25x^2 + 318x^3 + 530x^4,$$
$$\mu^r_{K_{4,4}}(x) = 3x + 694x^2 + 15695x^3 + 30264x^4.$$

$$p_{K_{3,5}}(x) = 240x + 37584x^2 + 290880x^3 + 113664x^4,$$
$$\mu_{K_{3,5}}(x) = x + 33x^2 + 225x^3 + 105x^4,$$
$$\mu^r_{K_{3,5}}(x) = 10x + 1566x^2 + 12120x^3 + 4736x^4.$$

Ax.II.6 Complete graphs

Let K_n be the complete graph of order n, $n \geq 4$.

Case of order $n = 4$**:**

Because of $K_4 = W_4$, seen from Case of order $n = 4$ in Ax.II.3.

Case of order $n = 5$**:**

Orientable

$$\begin{aligned}
p_{K_5}(x) &= 462x + 4974x^2 + 3240x^3, \\
\mu_{K_5}(x) &= 6x + 31x^2 + 13x^3, \\
\mu^r_{K_5}(x) &= 77x + 829x^2 + 390x^3.
\end{aligned}$$

Nonorientable

$$\begin{aligned}
q_{K_5}(x^{-1}) &= 54x + 1320x^2 + 17490x^3 + 84660x^4 \\
&\quad + +208776x^5 + 177588x^6, \\
\nu_{K_5}(x^{-1}) &= 2x + 11x^2 + 99x^3 + 417x^4 + 955x^5 + 796x^6, \\
\nu^r_{K_5}(x^{-1}) &= 9x + 220x^2 + 2915x^3 + 14110x^4 + 34796x^5 \\
&\quad + +29598x^6.
\end{aligned}$$

Case of order $n = 6$**:**

Orientable

$$\begin{aligned}
p_{K_6}(x) &= 1800x + 654576x^2 + 24613800X^3 \\
&\quad + +124250208x^4 + 41582592x^5.
\end{aligned}$$

Nonorientable

$$
\begin{aligned}
q_{K_6}(x^{-1}) = 24x &+ 4560x^2 + 370920x^3 + 10828440x^4 \\
&+ 192264576x^5 + 1927543032x^6 + 11905560960x^7 \\
&+ 42386101920x^8 + 79831388160x^9 \\
&+ 59244281856x^{10}.
\end{aligned}
$$

Appendix III

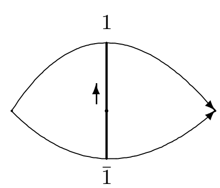
Atlas of Rooted and Unrooted Maps for Small Graphs

In the symbol $X : a, b, c$ for a map appearing under a figure below, X is the under graph of the map, $a = oy$ or qy are, respectively, orientable or nonorientable genus y, b is the series number with two digits and c is the number of ways to assign a root. And, $\bar{x}$ on a surface is for x^{-1}, or $-x$, $x = 1, 2, \cdots$.

Ax.III.1 Bouquets B_m of size $4 \geq m \geq 1$

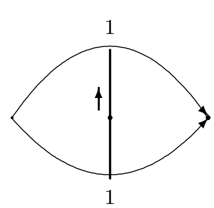
Case $m = 1$**:**

Orientable genus 0



$B_1 : o0 - 01 - 01$

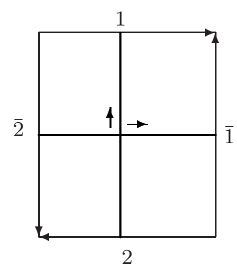
Nonorientable genus 1



$B_1 : q1 - 01 - 01$

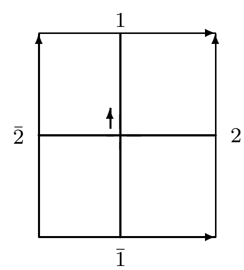
Case $m = 2$**:**

Orientable genus 0



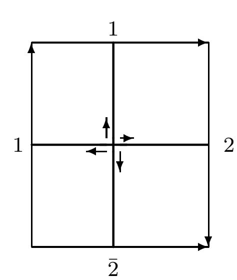
$B_2 : o0 - 01 - 02$

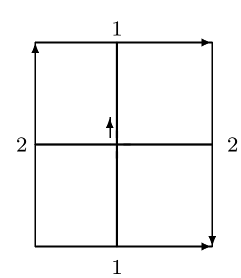
Orientable genus 1



$B_2 : o1 - 01 - 01$

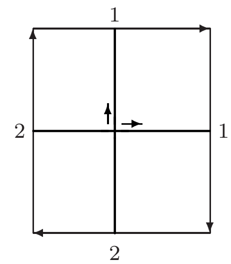
Nonorientable genus 1



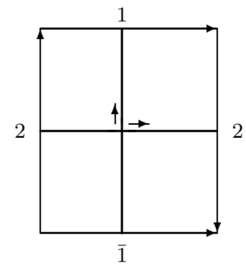


$B_2 : q1 - 01 - 04$ $B_2 : q1 - 02 - 01$

Nonorientable genus 2



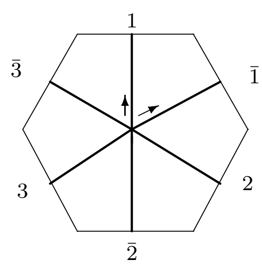
$B_2 : q2 - 01 - 02$



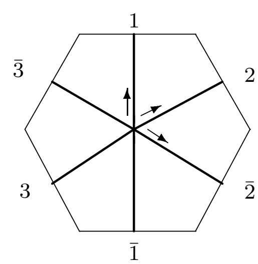
$B_2 : q2 - 02 - 02$

Case $m = 3$**:**

Orientable genus 0

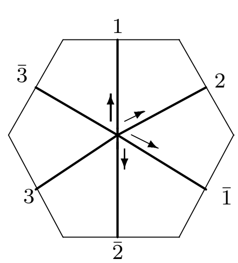


$B_3 : o0 - 01 - 02$

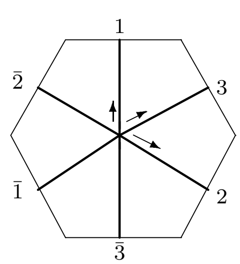


$B_3 : o0 - 02 - 03$

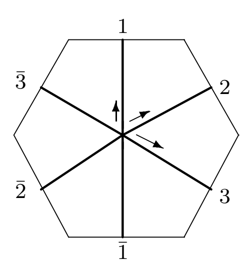
Orientable genus 1



$B_3 : o1 - 01 - 04$

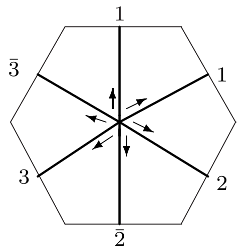


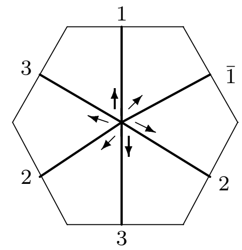
$B_3 : o1 - 02 - 03$

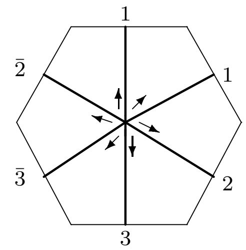


$B_3 : o1 - 03 - 03$

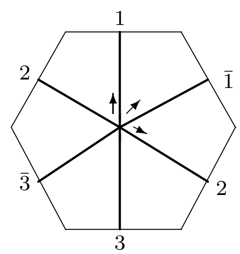
Nonorientable genus 1

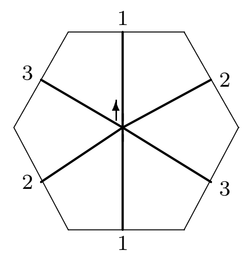






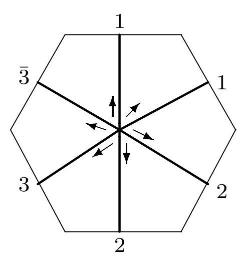
$B_3 : q1-01-06$ $B_3 : q1-02-06$ $B_3 : q1-03-06$

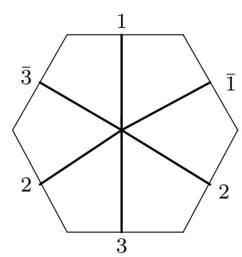


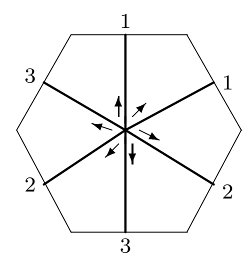


$B_3 : q1-04-03$ $B_3 : q1-05-01$

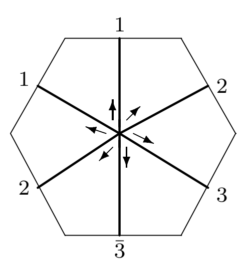
Nonorientable genus 2

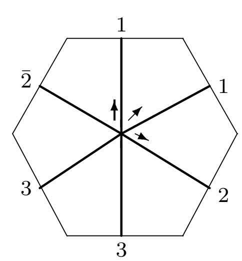


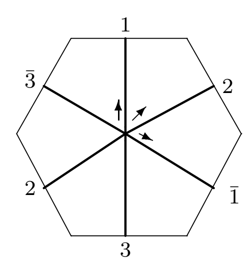




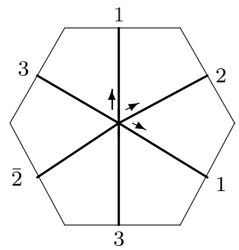
$B_3 : q2-01-06$ $B_3 : q2-02-12$ $B_3 : q2-03-06$



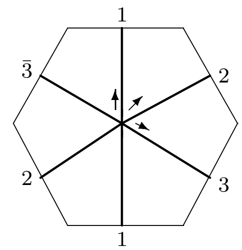




$B_3 : q2-04-06$ $B_3 : q2-05-03$ $B_3 : q2-06-03$

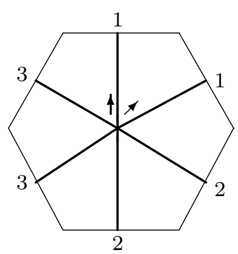


$B_3 : q2 - 07 - 03$

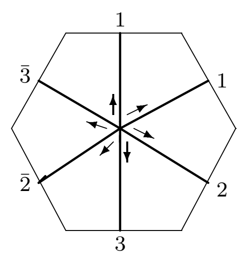


$B_3 : q2 - 08 - 03$

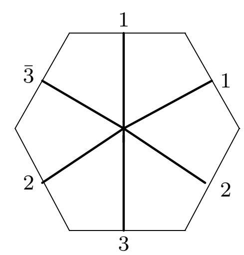
Nonorientable genus 3



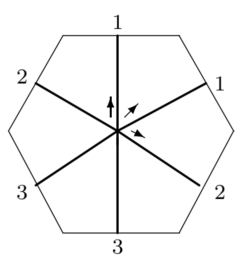
$B_3 : q3 - 01 - 02$



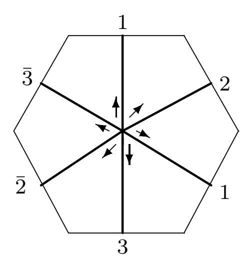
$B_3 : q3 - 02 - 06$



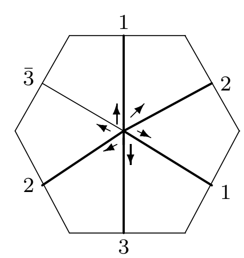
$B_3 : q3 - 03 - 12$



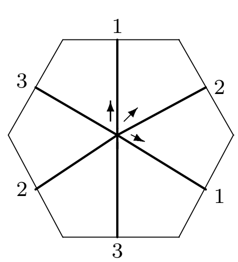
$B_3 : q3 - 04 - 03$



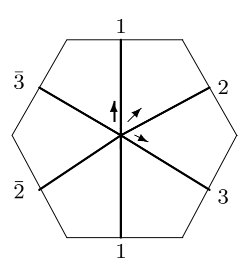
$B_3 : q3 - 05 - 06$



$B_3 : q3 - 06 - 06$



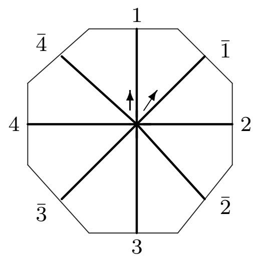
$B_3 : q3 - 07 - 03$

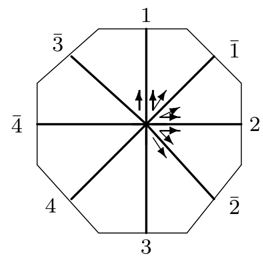


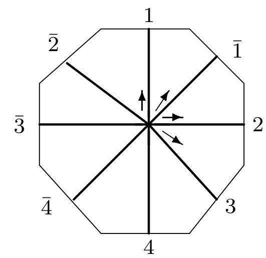
$B_3 : q3 - 08 - 03$

Case $m = 4$:

Orientable genus 0

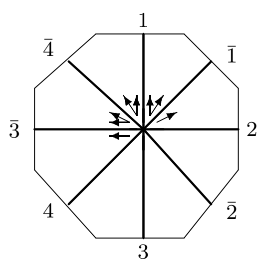


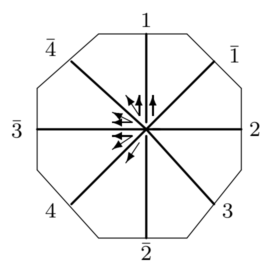


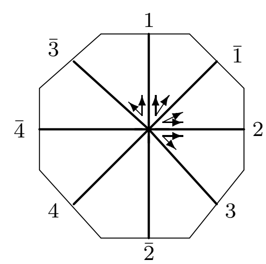


$B_4 : o0-01-02$ $B_4 : o0-02-08$ $B_4 : o0-03-04$

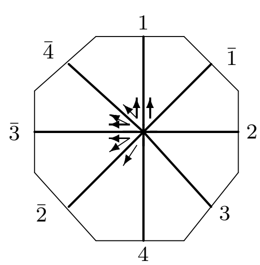
Orientable genus 1

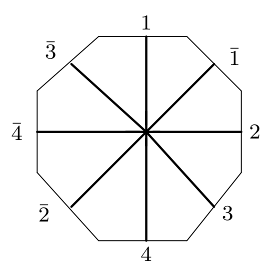


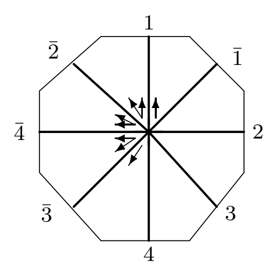




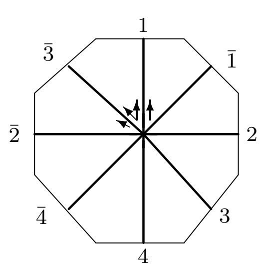
$B_4 : o1-01-08$ $B_4 : o1-02-08$ $B_4 : o1-03-08$

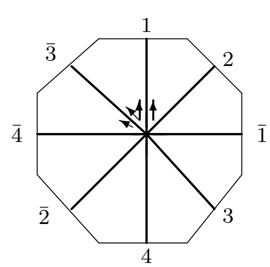


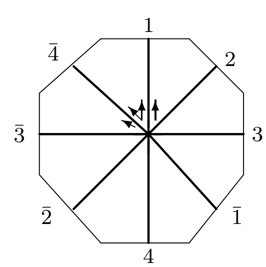




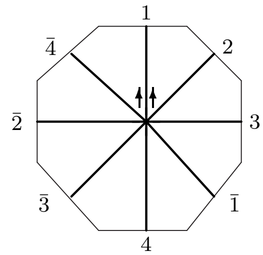
$B_4 : o1-04-08$ $B_4 : o1-05-16$ $B_4 : o1-06-08$





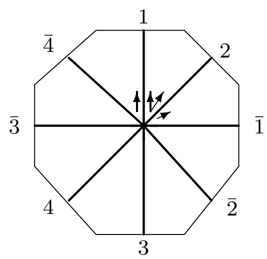


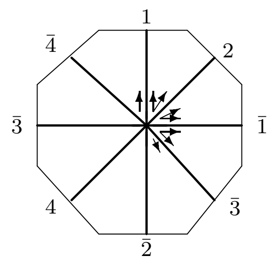
$B_4 : o1-07-04$ $B_4 : o1-08-04$ $B_4 : o1-09-04$

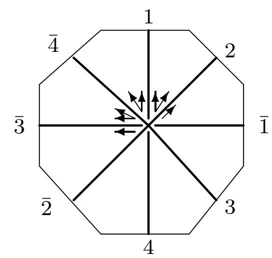


$B_4 : o1 - 10 - 02$

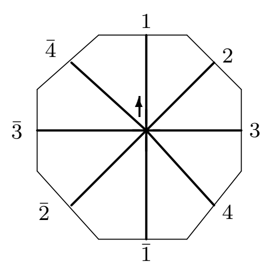
Orientable genus 2







$B_4 : o2 - 01 - 04$ $B_4 : o2 - 02 - 08$ $B_4 : o2 - 03 - 08$

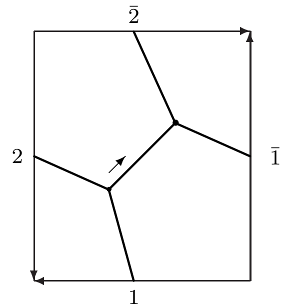


$B_4 : o2 - 04 - 01$

Ax.III.2 Link bundles $L_m, 6 \geq m \geq 3$

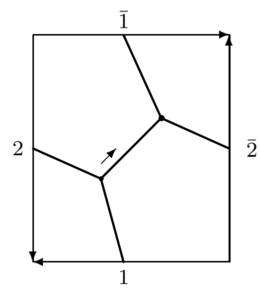
Case $m = 3$**:**

Orientable genus 0



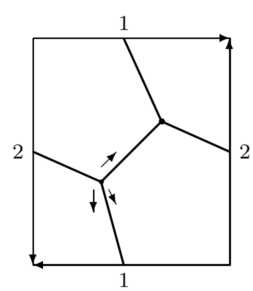
$L_3 : o0-01-01$

Orientable genus 1



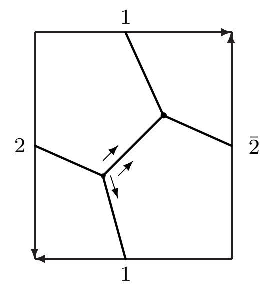
$L_3 : o1-01-01$

Nonorientable genus 1



$L_3 : q1-01-03$

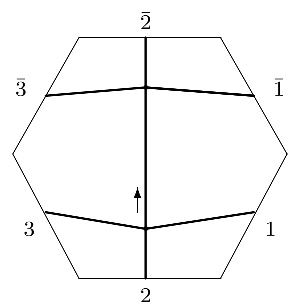
Nonorientable genus 2



$L_3 : q2 - 01 - 03$

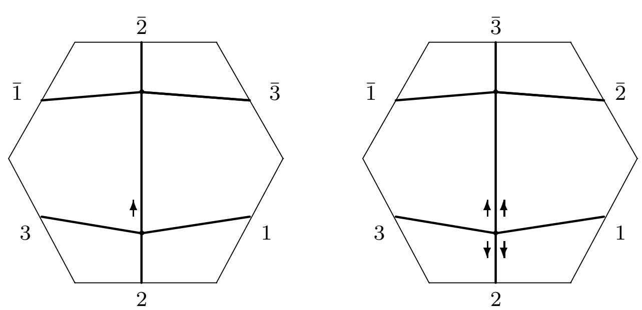
Case $m = 4$**:**

Orientable genus 0



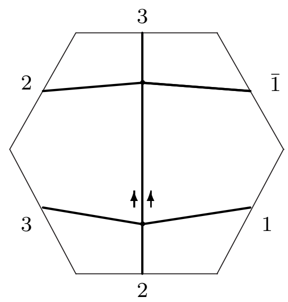
$L_4 : o0 - 01 - 01$

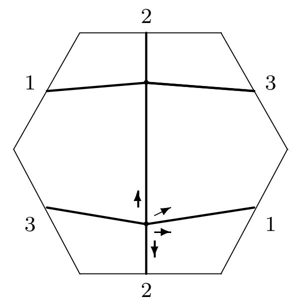
Orientable genus 1



$L_4 : o1 - 01 - 01$ $L_4 : o1 - 02 - 04$

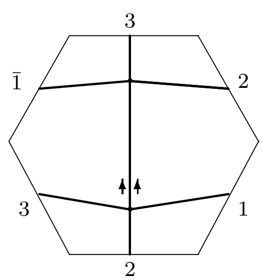
Nonorientable genus 1

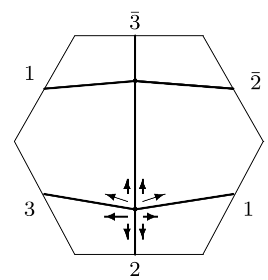


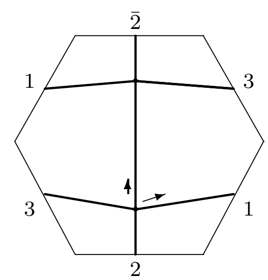


$L_4 : q1-01-02$ $L_4 : q1-02-04$

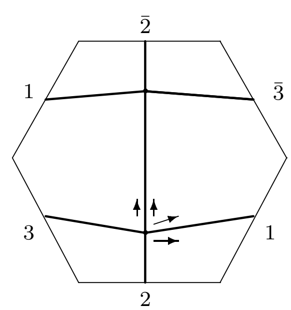
Nonorientable genus 2





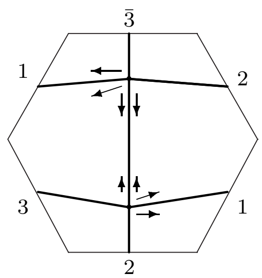


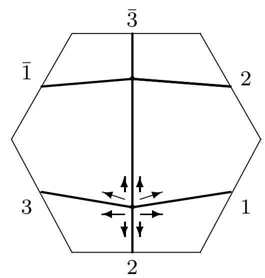
$L_4 : q2-01-02$ $L_4 : q2-02-08$ $L_4 : q2-03-02$

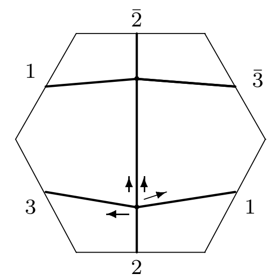


$L_4 : q2-04-04$

Nonorientable genus 3



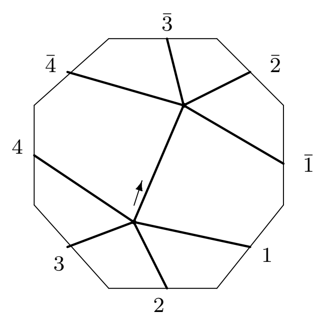




$L_4 : q3-01-08$ $L_4 : q3-02-08$ $L_4 : q3-03-04$

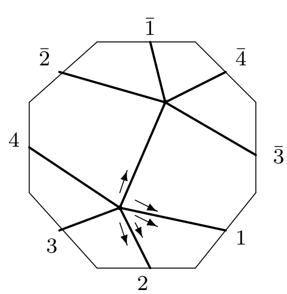
Case $m = 5$**:**

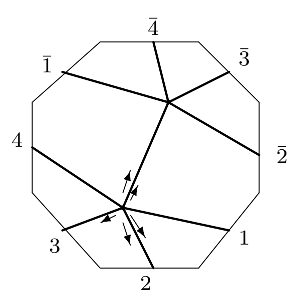
Orientable genus 0

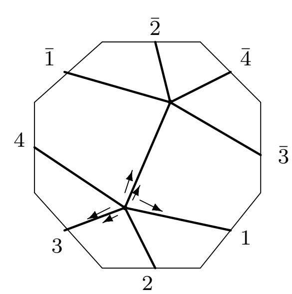


$L_5 : o0-01-01$

Orientable genus 1

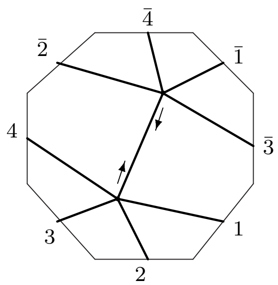




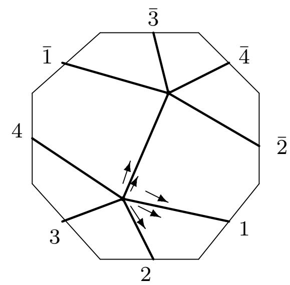


$L_5 : o1-01-05$ $L_5 : o1-02-05$ $L_5 : o1-03-05$

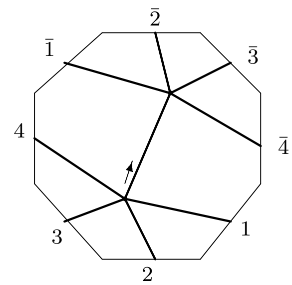
Orientable genus 2



$L_5 : o2 - 01 - 02$

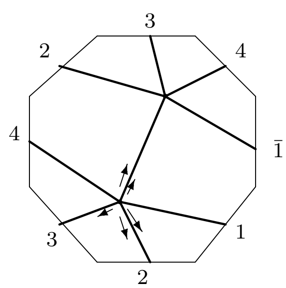


$L_5 : o2 - 02 - 05$

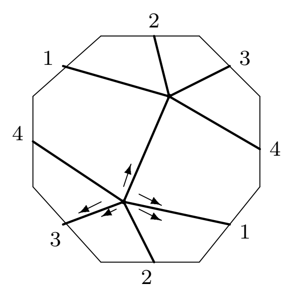


$L_5 : o2 - 03 - 01$

Nonorientable genus 1

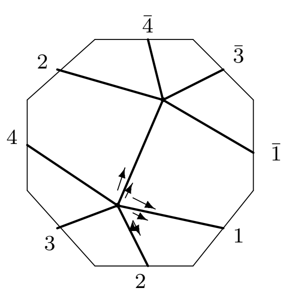


$L_5 : q1 - 01 - 05$

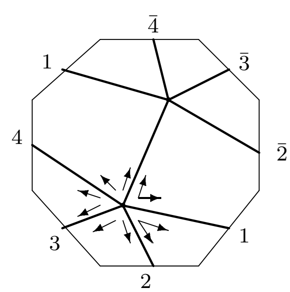


$L_5 : q1 - 02 - 05$

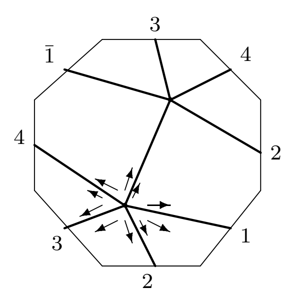
Nonorientable genus 2



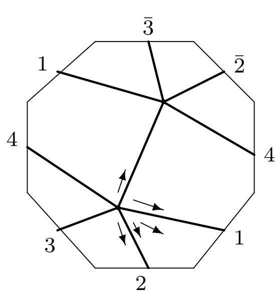
$L_5 : q2 - 01 - 05$



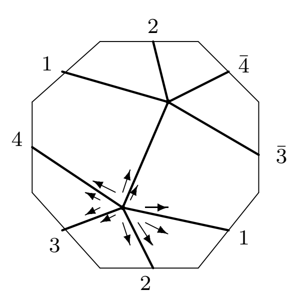
$L_5 : q2 - 02 - 10$



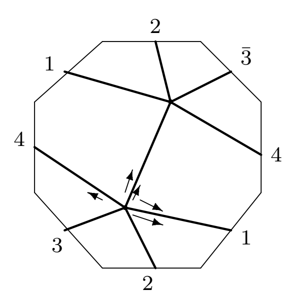
$L_5 : q2 - 03 - 10$



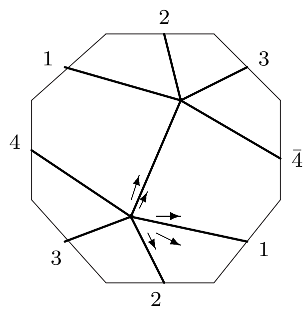
$L_5 : q2 - 04 - 05$



$L_5 : q2 - 05 - 10$

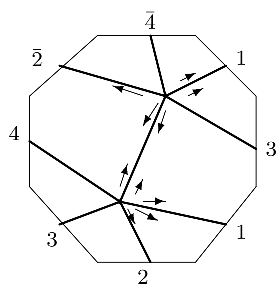


$L_5 : q2 - 06 - 05$

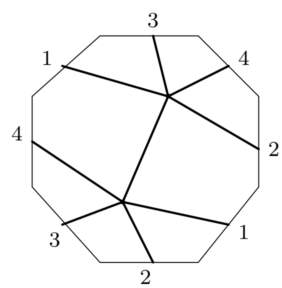


$L_5 : q2 - 07 - 05$

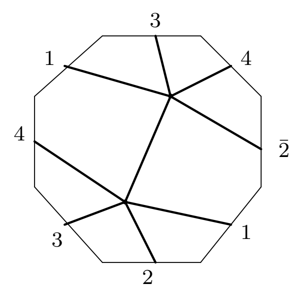
Nonorientable genus 3



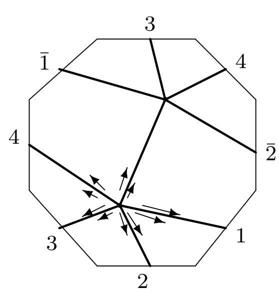
$L_5 : q3 - 01 - 10$



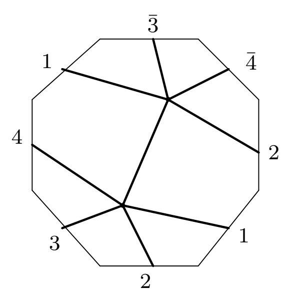
$L_5 : q3 - 02 - 20$



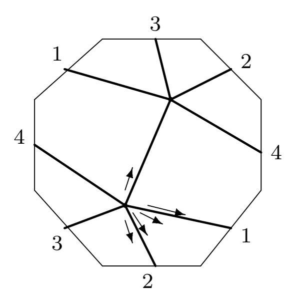
$L_5 : q3 - 03 - 20$



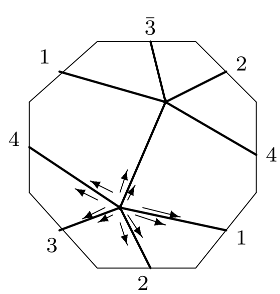
$L_5 : q3 - 04 - 10$



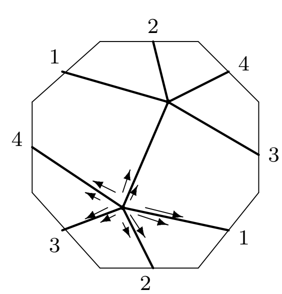
$L_5 : q3 - 05 - 20$



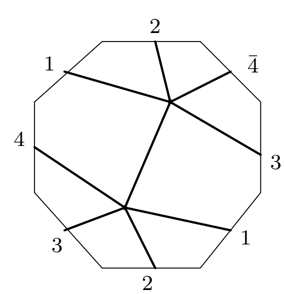
$L_5 : q3 - 06 - 05$



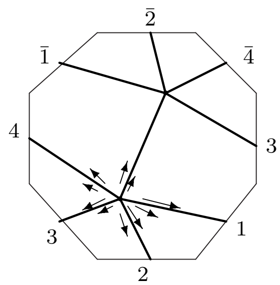
$L_5 : q3 - 07 - 10$

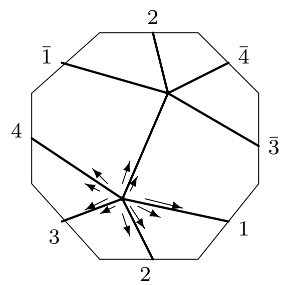


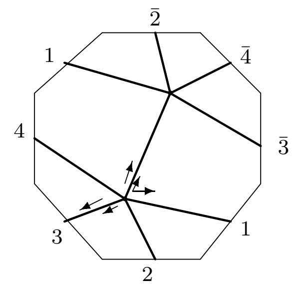
$L_5 : q3 - 08 - 10$



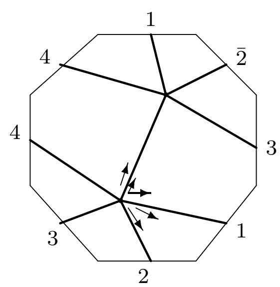
$L_5 : q3 - 09 - 20$

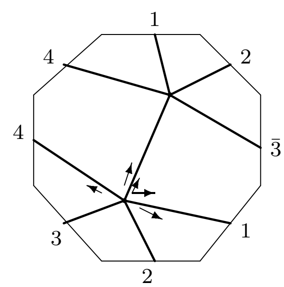






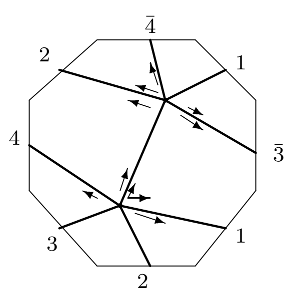
$L_5 : q3-10-10$ $L_5 : q3-11-10$ $L_5 : q3-12-05$

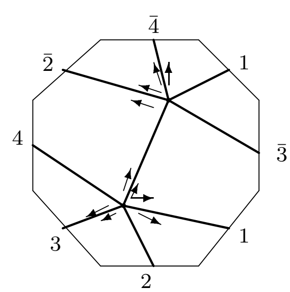


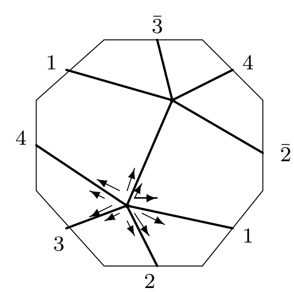


$L_5 : q3-13-05$ $L_5 : q3-14-05$

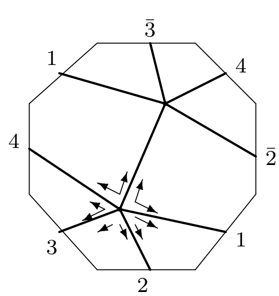
Nonorientable genus 4

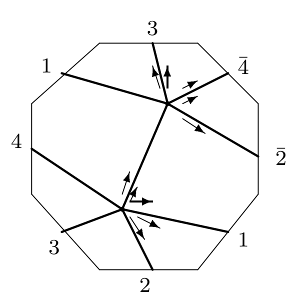


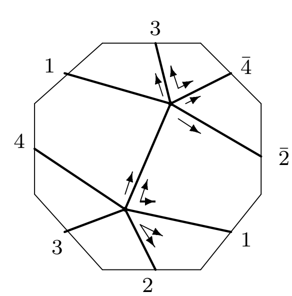




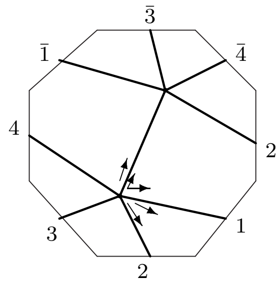
$L_5 : q4-01-10$ $L_5 : q4-02-10$ $L_5 : q4-03-10$



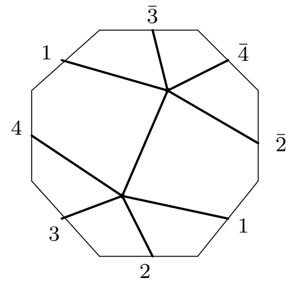




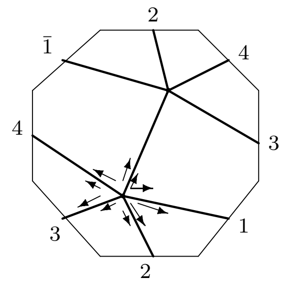
$L_5 : q4-04-10$ $L_5 : q4-05-10$ $L_5 : q4-06-10$



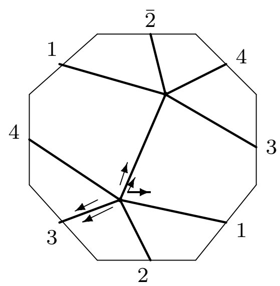
$L_5 : q4 - 07 - 05$



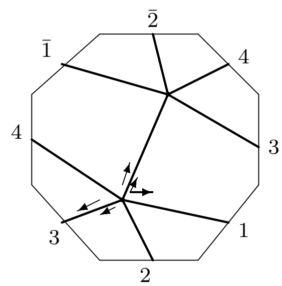
$L_5 : q4 - 08 - 20$



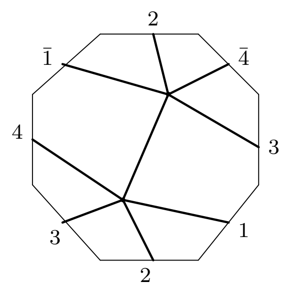
$L_5 : q4 - 09 - 10$



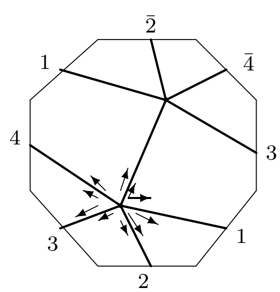
$L_5 : q4 - 10 - 05$



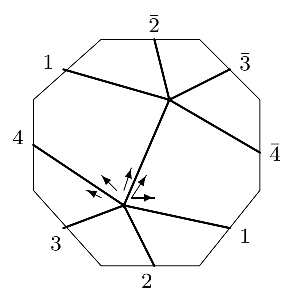
$L_5 : q4 - 11 - 05$



$L_5 : q4 - 12 - 20$



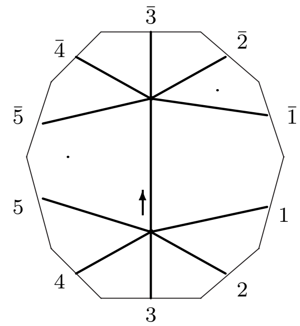
$L_5 : q4 - 13 - 10$



$L_5 : q4 - 14 - 05$

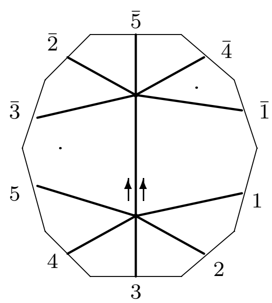
Case $m = 6$:

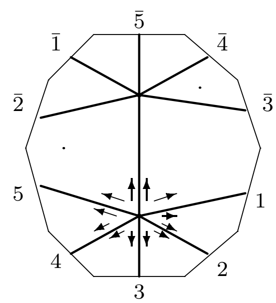
Orientable genus 0

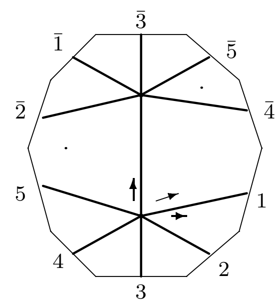


$L_6 : o0 - 01 - 01$

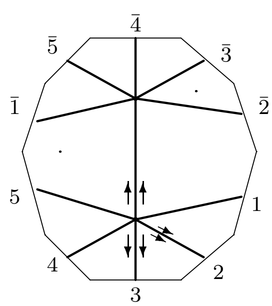
Orientable genus 1

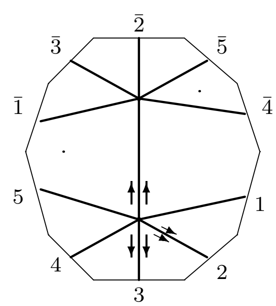


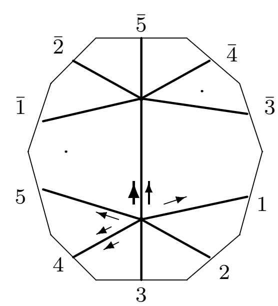




$L_6 : o1 - 01 - 02$ $L_6 : o1 - 02 - 12$ $L_6 : o1 - 03 - 03$

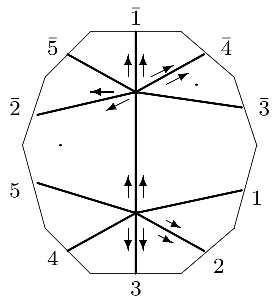




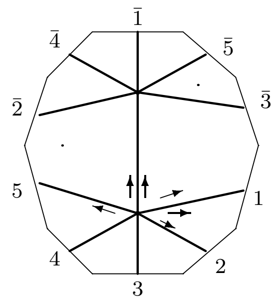


$L_6 : o1 - 04 - 06$ $L_6 : o1 - 05 - 06$ $L_6 : o1 - 06 - 06$

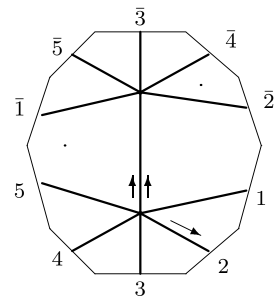
Orientable genus 2



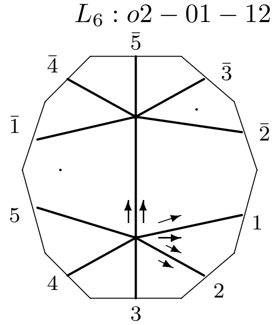
$L_6 : o2 - 01 - 12$



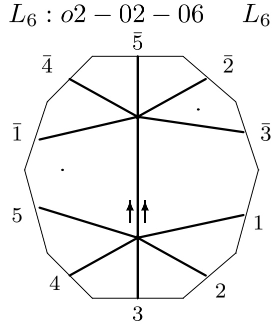
$L_6 : o2 - 02 - 06$



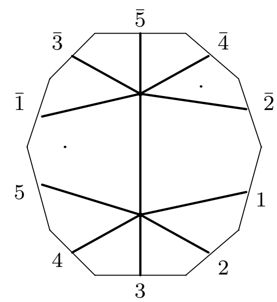
$L_6 : o2 - 03 - 03$



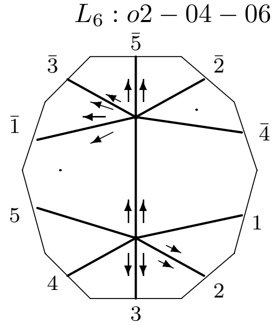
$L_6 : o2 - 04 - 06$



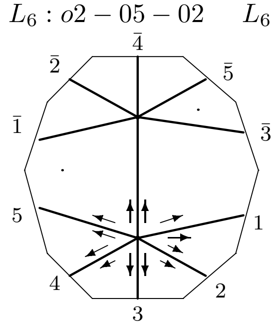
$L_6 : o2 - 05 - 02$



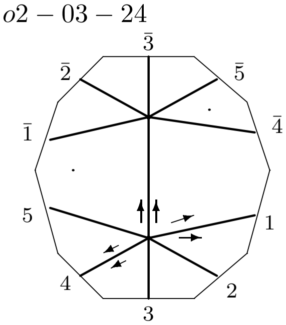
$L_6 : o2 - 03 - 24$



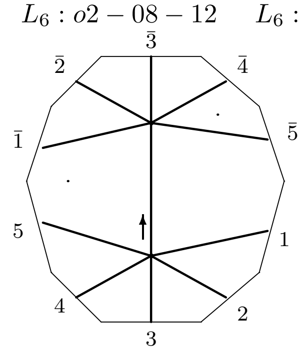
$L_6 : o2 - 07 - 12$



$L_6 : o2 - 08 - 12$



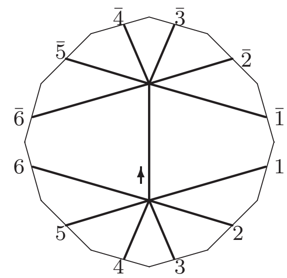
$L_6 : o2 - 09 - 06$



$L_6 : o2 - 01 - 01$

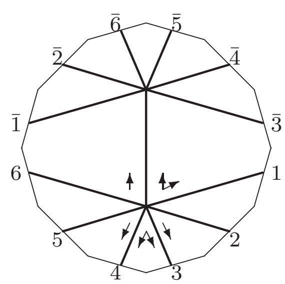
Case $m = 7$:

Orientable genus 0

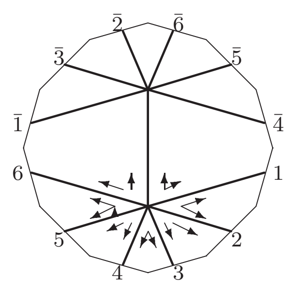


$L_7 : o0 - 01 - 01$

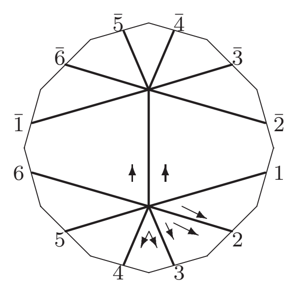
Orientable genus 1



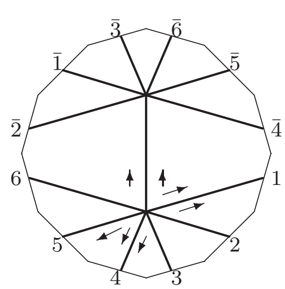
$L_7 : o1 - 01 - 07$



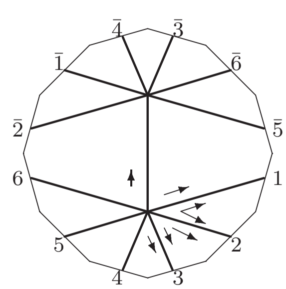
$L_7 : o1 - 02 - 14$



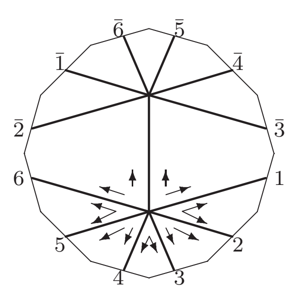
$L_7 : o1 - 03 - 07$



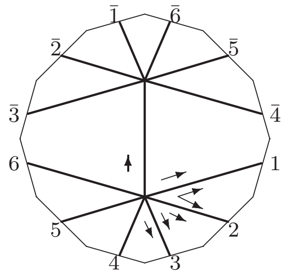
$L_7 : o1 - 04 - 07$

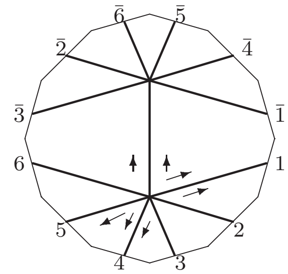


$L_7 : o1 - 05 - 07$



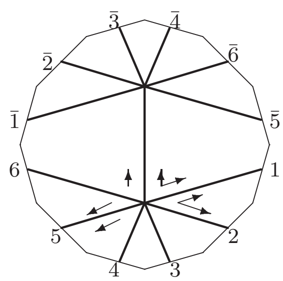
$L_7 : o1 - 06 - 14$

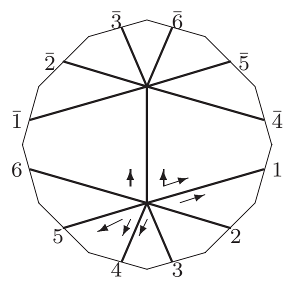


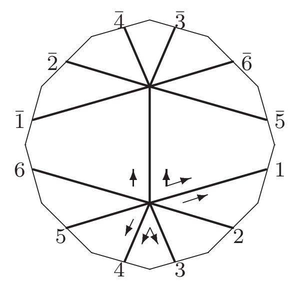


$L_7 : o1 - 07 - 07$ $L_7 : o1 - 08 - 07$

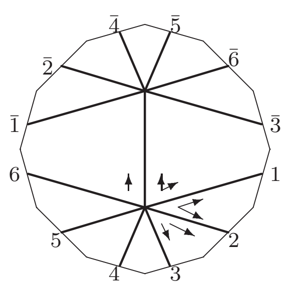
Orientable genus 2

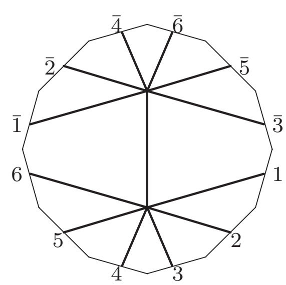


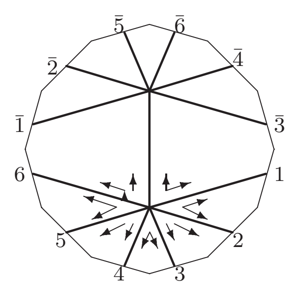




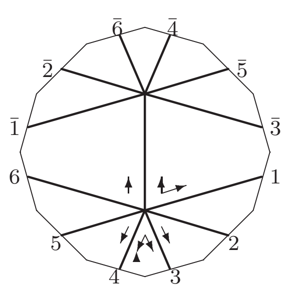
$L_7 : o2 - 01 - 07$ $L_7 : o2 - 02 - 07$ $L_7 : o2 - 03 - 07$

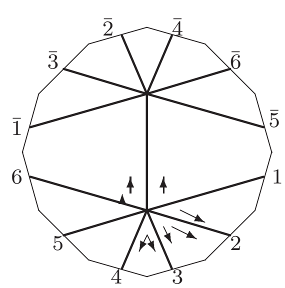


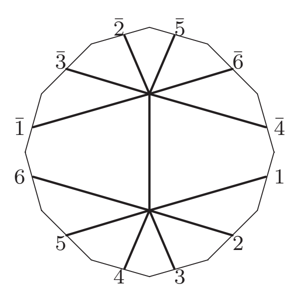




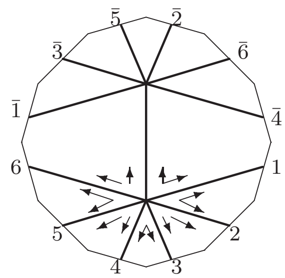
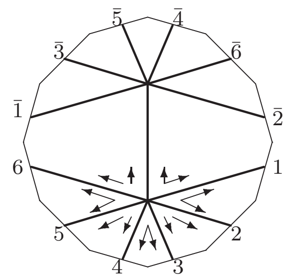
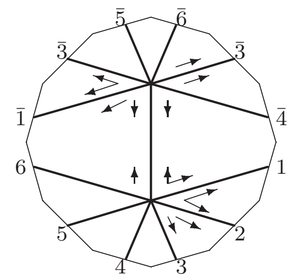
$L_7 : o2 - 04 - 07$ $L_7 : o2 - 05 - 28$ $L_7 : o2 - 06 - 14$



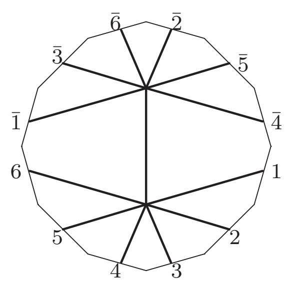
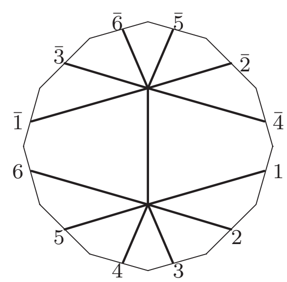
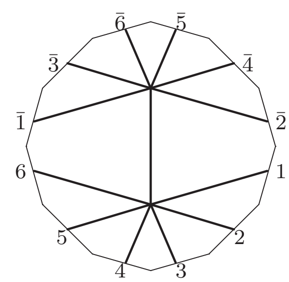




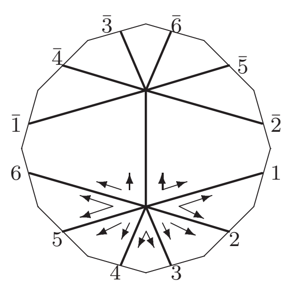
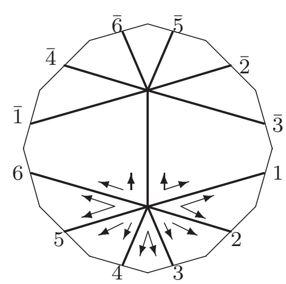
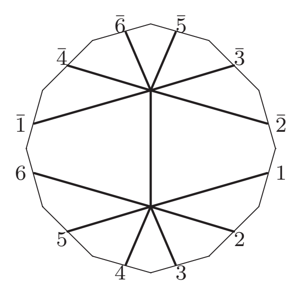
$L_7 : o2 - 07 - 07$ $L_7 : o2 - 08 - 07$ $L_7 : o2 - 09 - 28$

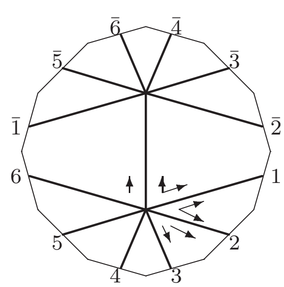
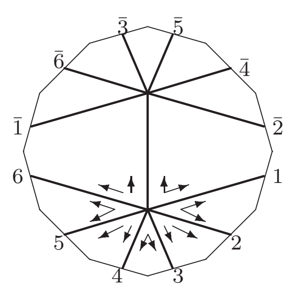
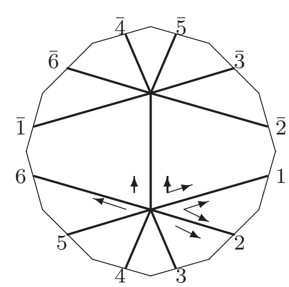
$L_7 : o2-10-14$ $L_7 : o2-11-14$ $L_7 : o2-12-14$

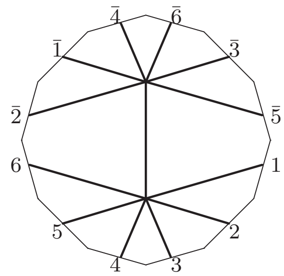
$L_7 : o2-13-28$ $L_7 : o2-14-28$ $L_7 : o2-15-28$

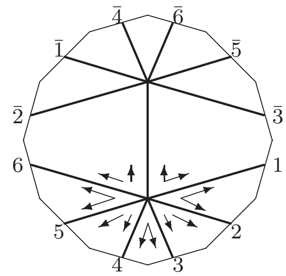
$L_7 : o2-16-14$ $L_7 : o2-17-14$ $L_7 : o2-18-28$

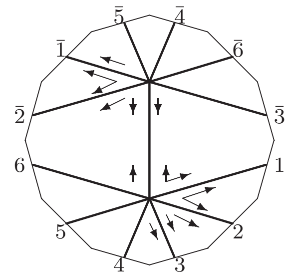
$L_7 : o2-19-07$ $L_7 : o2-20-14$ $L_7 : o2-21-07$



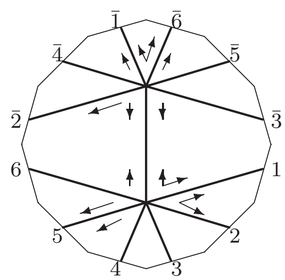
$L_7 : o2-22-28$



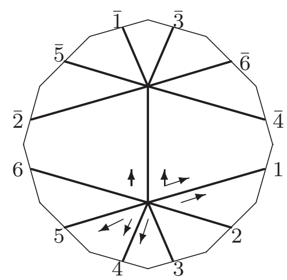
$L_7 : o2-23-14$



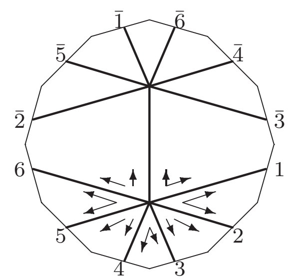
$L_7 : o2-24-14$



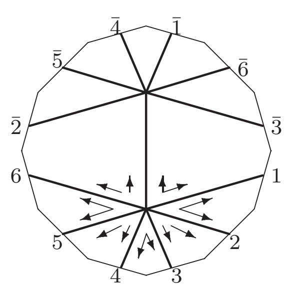
$L_7 : o2-25-14$



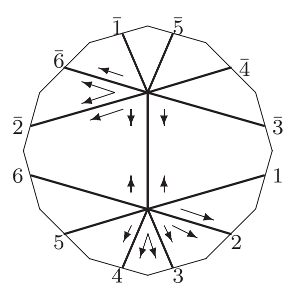
$L_7 : o2-26-07$



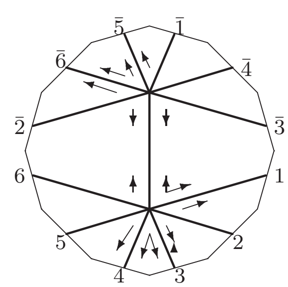
$L_7 : o2-27-14$



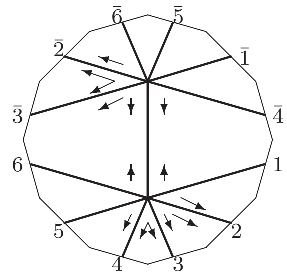
$L_7 : o2-28-14$



$L_7 : o2-29-14$

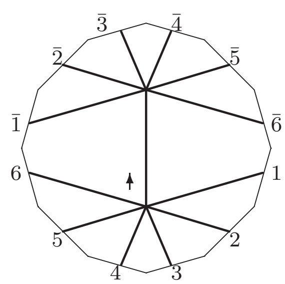


$L_7 : o2-30-14$

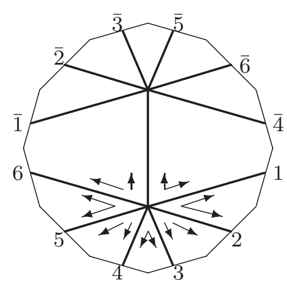


$L_7 : o2-31-14$

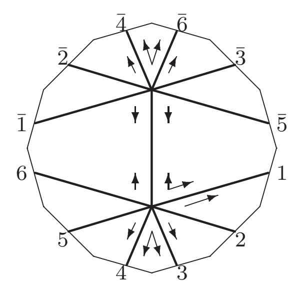
Orientable genus 3



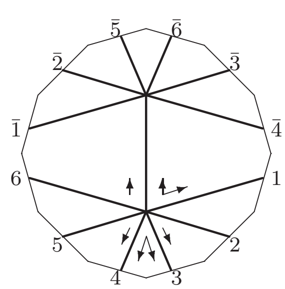
$L_7 : o3-01-01$



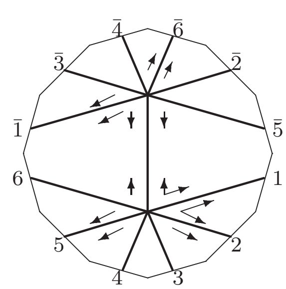
$L_7 : o3-02-14$



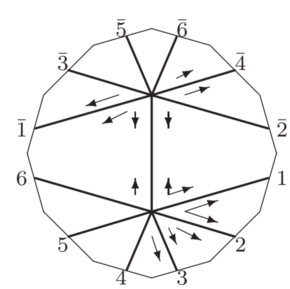
$L_7 : o3-03-14$



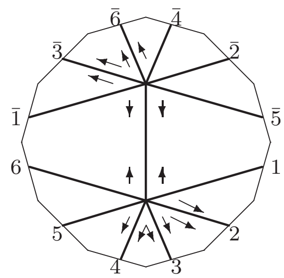
$L_7 : o3-04-07$



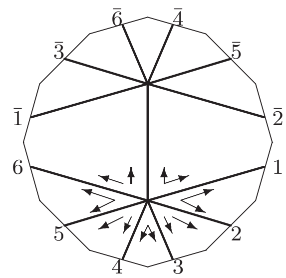
$L_7 : o3-05-14$



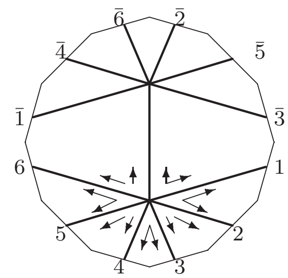
$L_7 : o3-06-14$



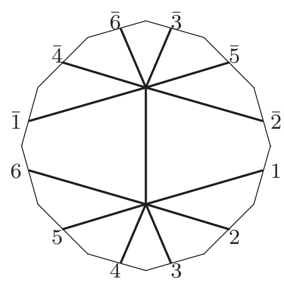
$L_7 : o3 - 07 - 14$



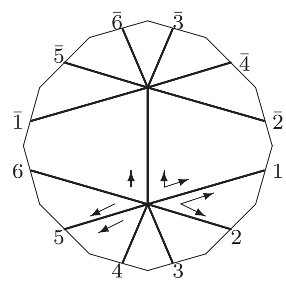
$L_7 : o3 - 08 - 14$



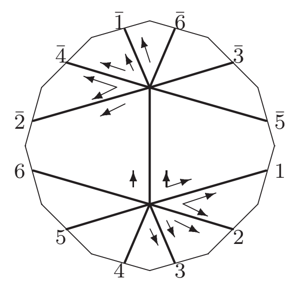
$L_7 : o3 - 09 - 14$



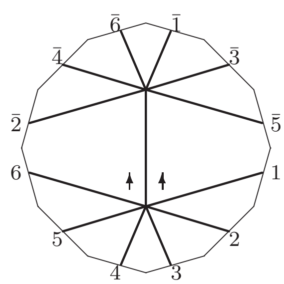
$L_7 : o3 - 10 - 28$



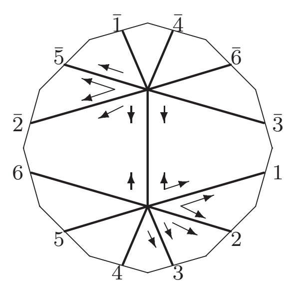
$L_7 : o3 - 11 - 07$



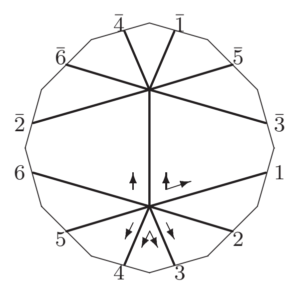
$L_7 : o3 - 12 - 14$



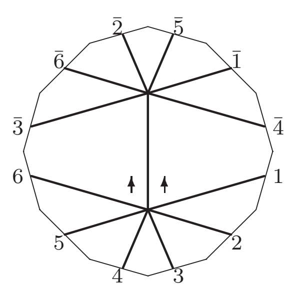
$L_7 : o3 - 13 - 02$



$L_7 : o3 - 14 - 14$



$L_7 : o3 - 15 - 07$

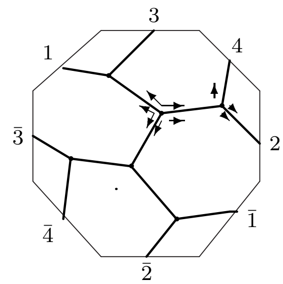


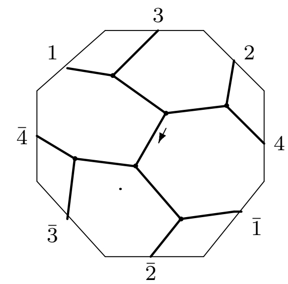
$L_7 : o3 - 16 - 02$

Ax.III.3 Complete bipartite graphs $K_{m,n}, 4 \geq m, n \geq 3$

Case $m + n = 6$**:**

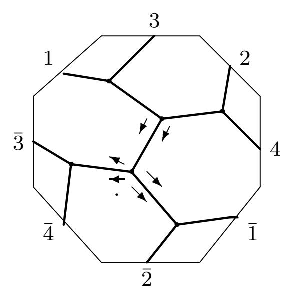
Orientable genus 1





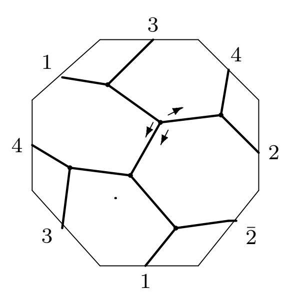
$K_{3,3} : o1 - 01 - 09$ $\qquad$ $K_{3,3} : o1 - 02 - 01$

Orientable genus 2



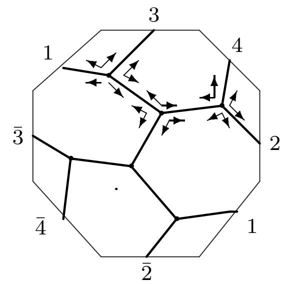
$K_{3,3} : o2 - 01 - 06$

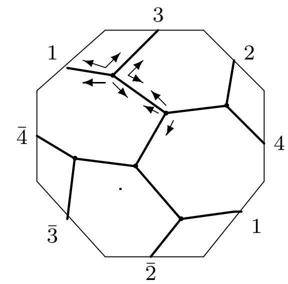
Nonorientable genus 1



$K_{3,3} : q1 - 01 - 03$

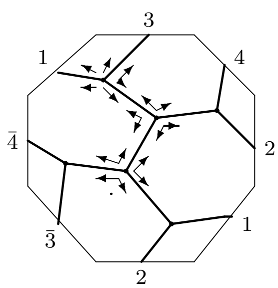
Nonorientable genus 2

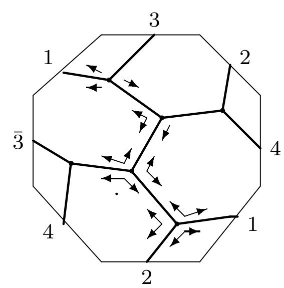


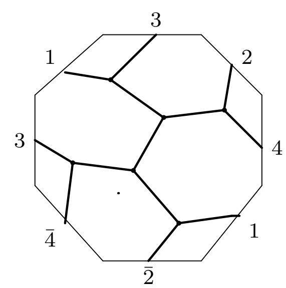


$K_{3,3} : q2 - 01 - 18$ $K_{3,3} : q2 - 02 - 09$

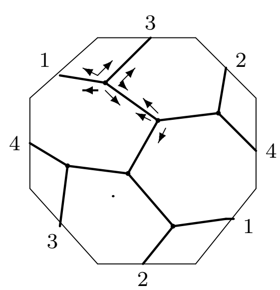
Nonorientable genus 3

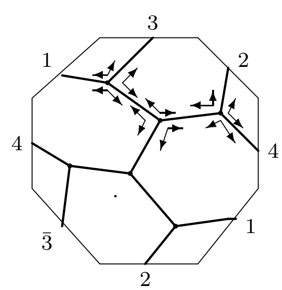


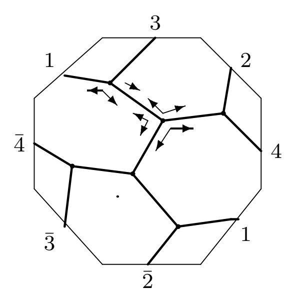




$K_{3,3} : q3 - 01 - 18$ $K_{3,3} : q3 - 02 - 18$ $K_{3,3} : q3 - 03 - 36$

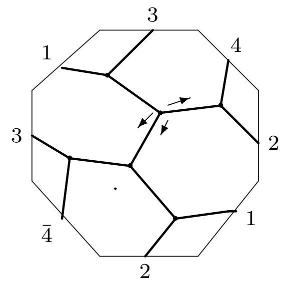




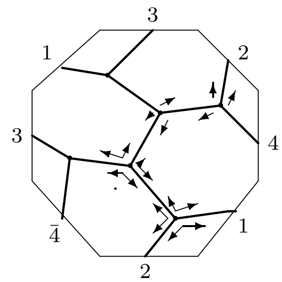


$K_{3,3} : q3 - 04 - 09$ $K_{3,3} : q3 - 05 - 18$ $K_{3,3} : q3 - 06 - 09$

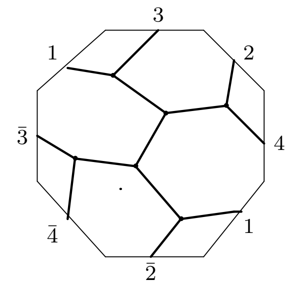
Nonorientable genus 4



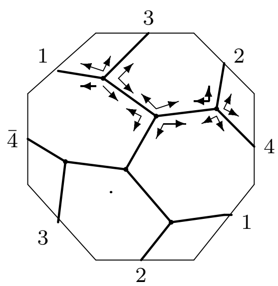
$K_{3,3} : q4-01-03$



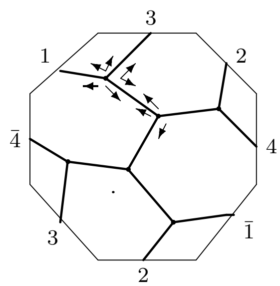
$K_{3,3} : q4-02-18$



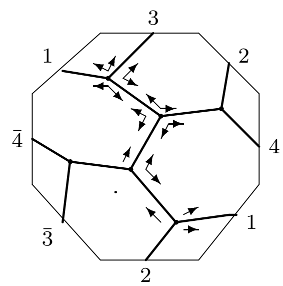
$K_{3,3} : q4-03-36$



$K_{3,3} : q4-04-18$



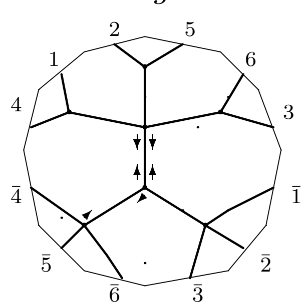
$K_{3,3} : q4-05-09$



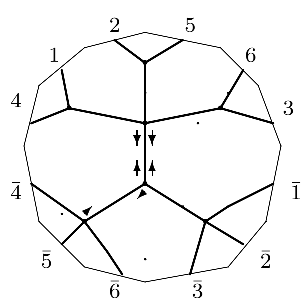
$K_{3,3} : q4-06-18$

Case $m+n=7$:

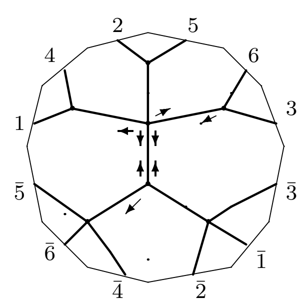
Orientable genus 1



$K_{4,3} : o1-01-06$

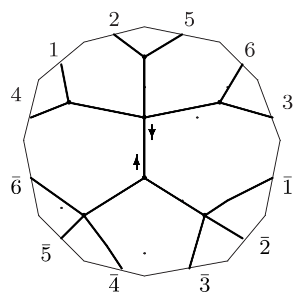


$K_{4,3} : o1-02-12$

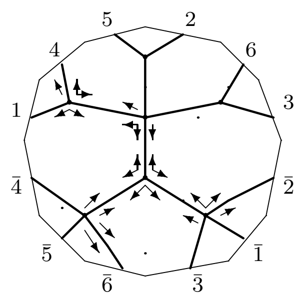


$K_{4,3} : o1-03-08$

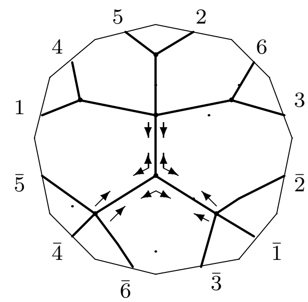
Orientable genus 2



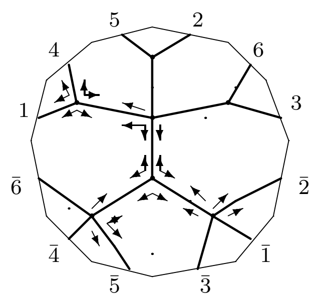
$K_{4,3} : o2-01-02$



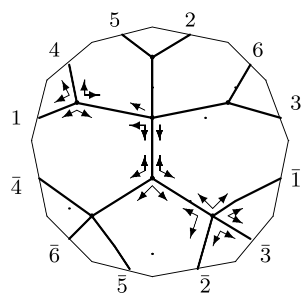
$K_{4,3} : o2-02-24$



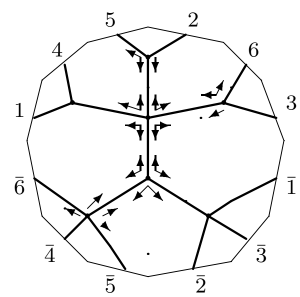
$K_{4,3} : o2-03-12$



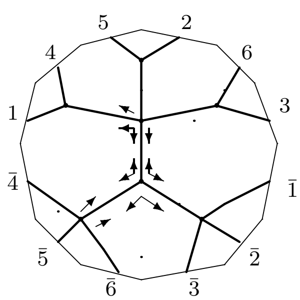
$K_{4,3} : o2-04-24$



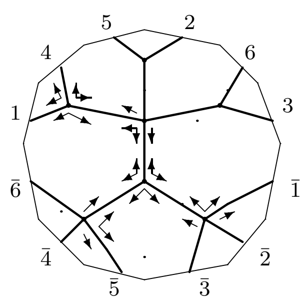
$K_{4,3} : o2-05-24$



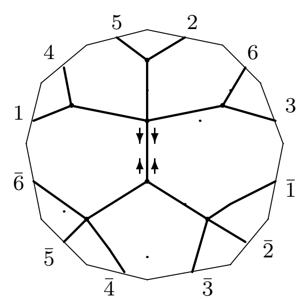
$K_{4,3} : o2-06-24$



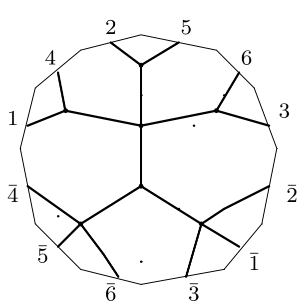
$K_{4,3} : o2-07-12$



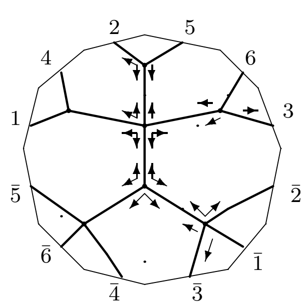
$K_{4,3} : o2-08-24$



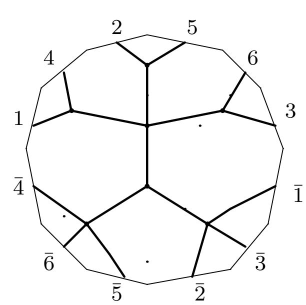
$K_{4,3} : o2-09-04$



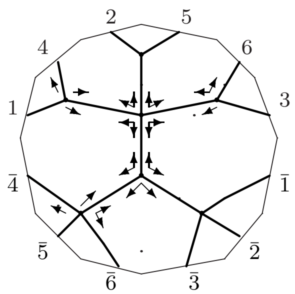
$K_{4,3} : o2-10-48$



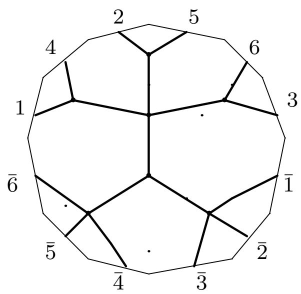
$K_{4,3} : o2-11-24$



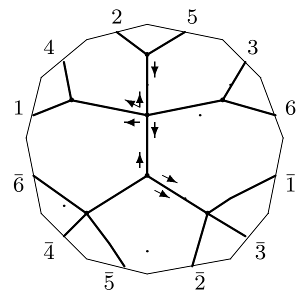
$K_{4,3} : o2-12-48$



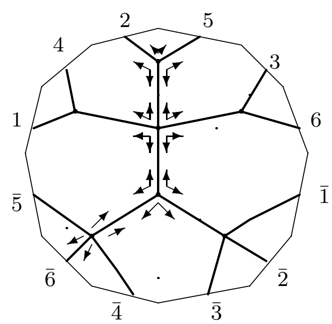
$K_{4,3} : o2-13-24$



$K_{4,3} : o2-14-48$

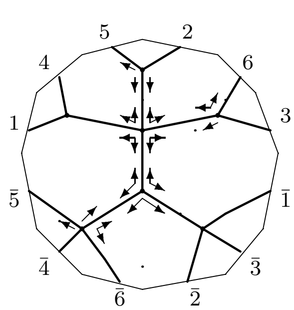


$K_{4,3} : o2-15-08$

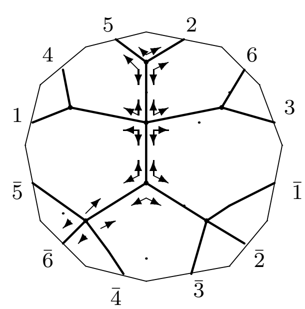


$K_{4,3} : o2-16-24$

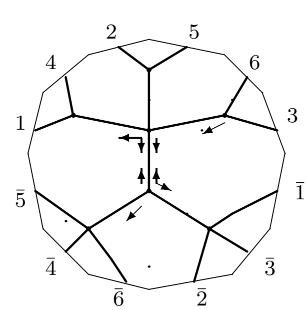
Orientable genus 3



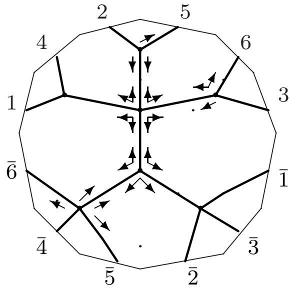
$K_{4,3} : o3-01-24$



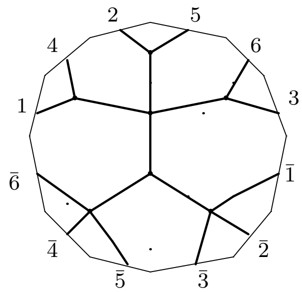
$K_{4,3} : o3-02-24$



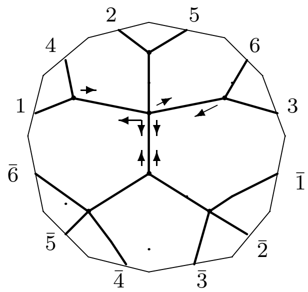
$K_{4,3} : o3-03-08$



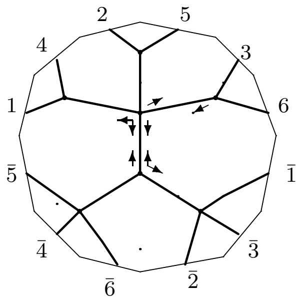
$K_{4,3} : o3 - 04 - 24$



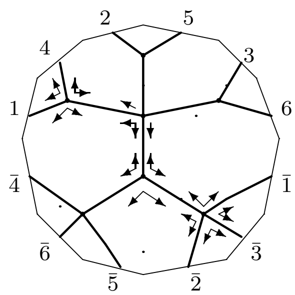
$K_{4,3} : o3 - 05 - 48$



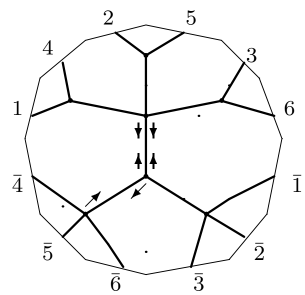
$K_{4,3} : o3 - 06 - 08$



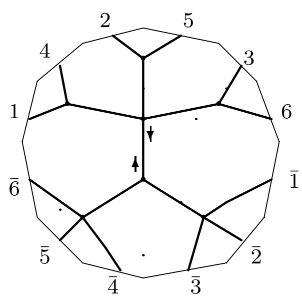
$K_{4,3} : o3 - 07 - 08$



$K_{4,3} : o3 - 08 - 24$



$K_{4,3} : o3 - 09 - 06$

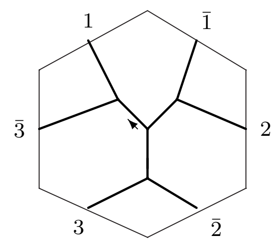


$K_{4,3} : o3 - 10 - 02$

Ax.III.4 Wheels $W_n, 5 \geq n \geq 4$

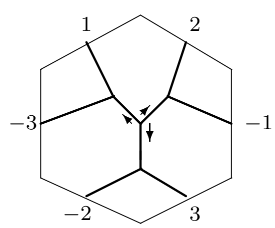
Case $n = 4$(*i.e.*, the complete graph K_4 of order 4):

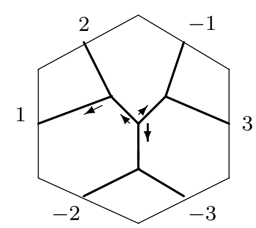
Orientable genus 0



$W_4 : o0 - 01 - 01$

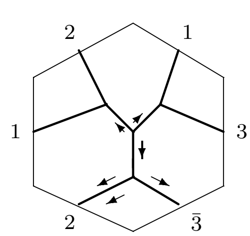
Orientable genus 1

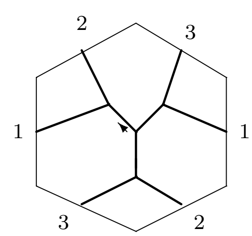




$W_4 : o1 - 01 - 03$ $W_4 : o1 - 02 - 04$

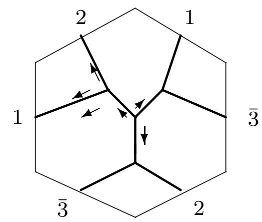
Nonorientable genus 1



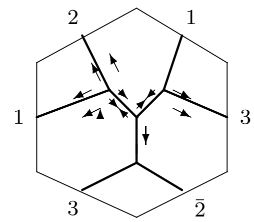


$W_4 : q1 - 01 - 06$ $W_4 : q1 - 02 - 01$

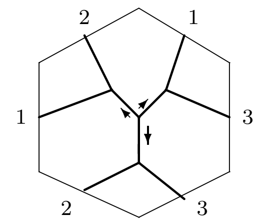
Nonorientable genus 2



$W_4 : q2 - 01 - 06$

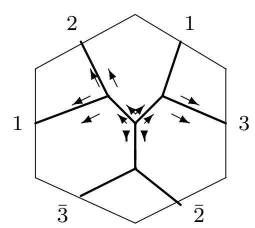


$W_4 : q2 - 02 - 12$

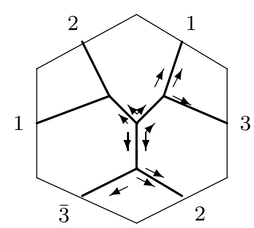


$W_4 : q2 - 03 - 03$

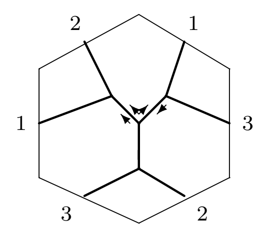
Nonorientable genus 3



$W_4 : q3 - 01 - 12$



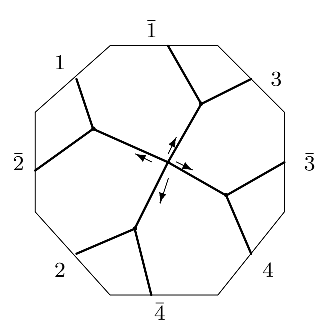
$W_4 : q3 - 02 - 12$



$W_4 : q3 - 03 - 04$

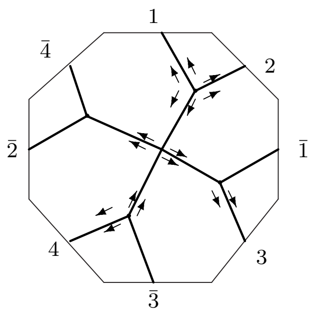
Case $n = 5$:

Orientable genus 0

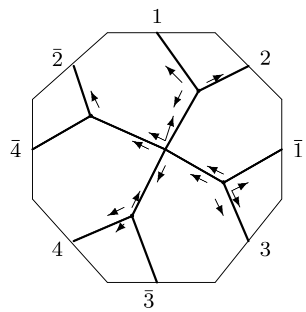


$W_5 : o0 - 01 - 04$

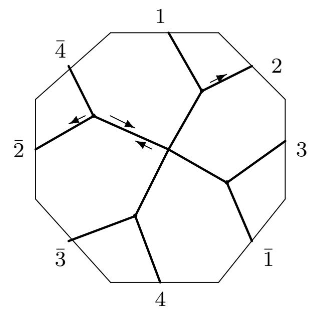
Orientable genus 1



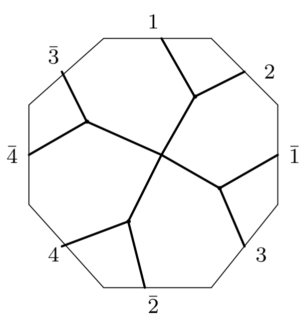
$W_5 : o1 - 01 - 16$



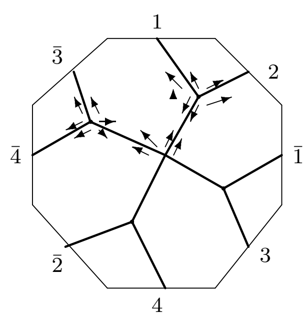
$W_5 : o1 - 02 - 16$



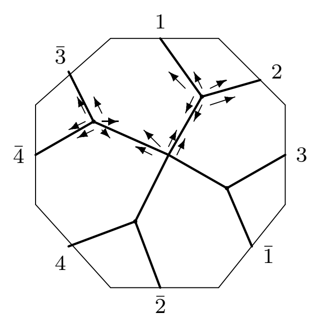
$W_5 : o1 - 03 - 04$



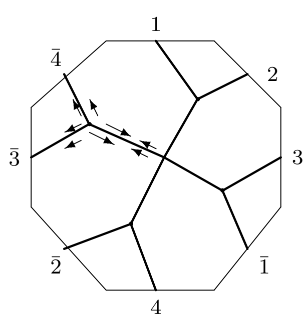
$W_5 : o1 - 04 - 32$



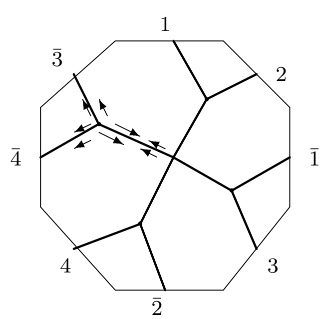
$W_5 : o1 - 05 - 16$



$W_5 : o1 - 06 - 16$

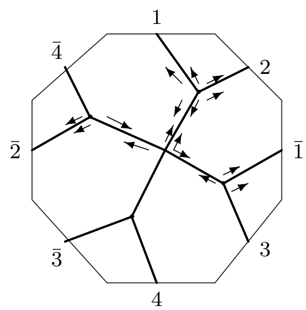


$W_5 : o1 - 07 - 08$

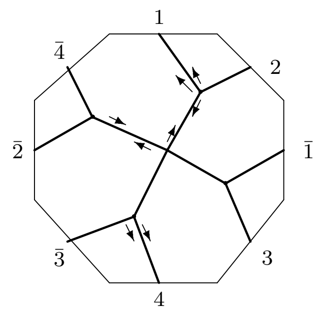


$W_5 : o1 - 08 - 08$

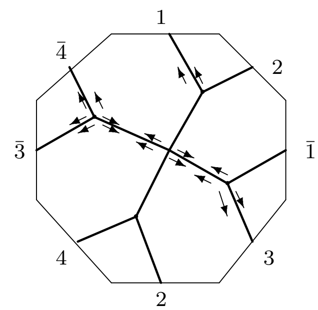
Orientable genus 2



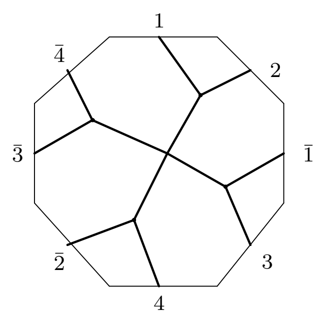
$W_5 : o2 - 01 - 16$



$W_5 : o2 - 02 - 08$



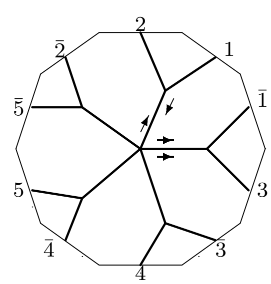
$W_5 : o2 - 03 - 16$



$W_5 : o2 - 04 - 32$

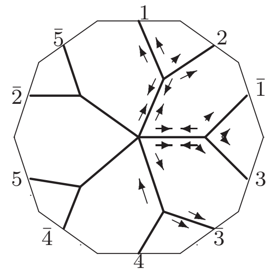
Case $n = 6$**:**

Orientable genus 0

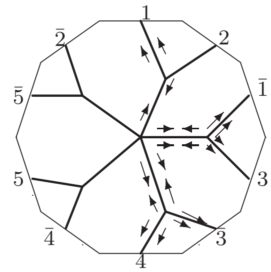


$W_6 : o0 - 01 - 04$

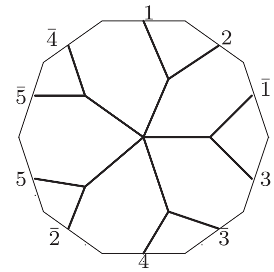
Orientable genus 1



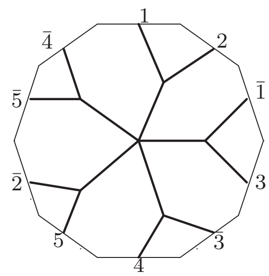
$W_6 : o1-01-20$



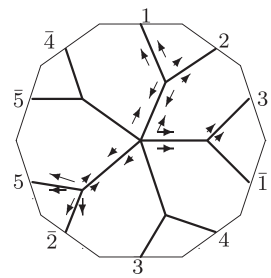
$W_6 : o1-02-20$



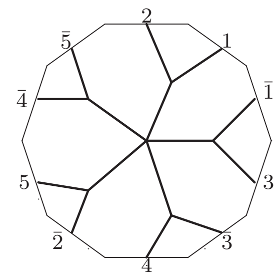
$W_6 : o1-03-40$



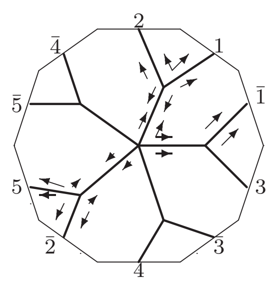
$W_6 : o1-04-40$



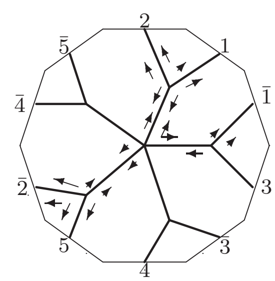
$W_6 : o1-05-20$



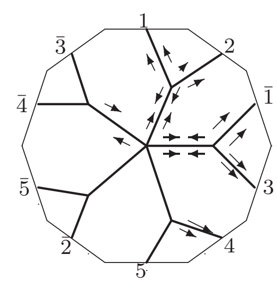
$W_6 : o1-06-40$



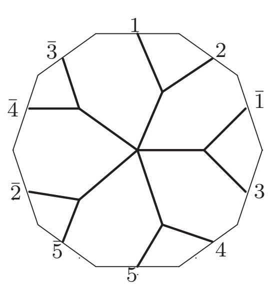
$W_6 : o1-07-20$



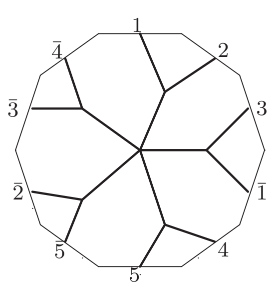
$W_6 : o1-08-20$



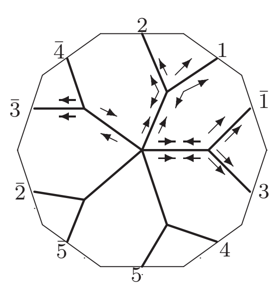
$W_6 : o1-09-20$



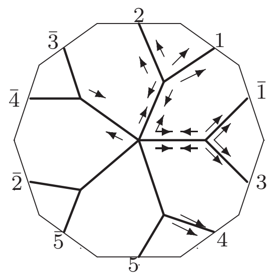
$W_6 : o1-10-40$



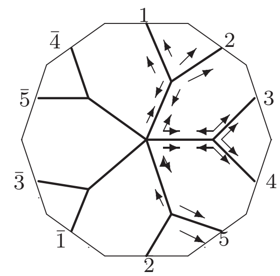
$W_6 : o1-11-40$



$W_6 : o1-12-20$

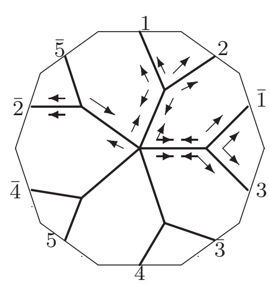


$W_6 : o1 - 13 - 20$

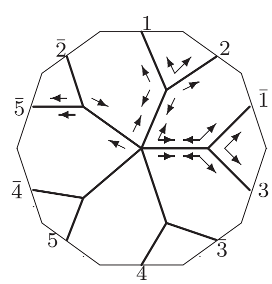


$W_6 : o1 - 14 - 20$

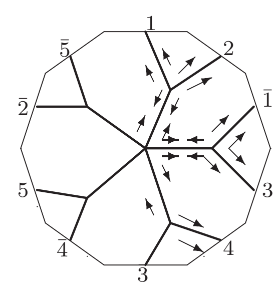
Orientable genus 2



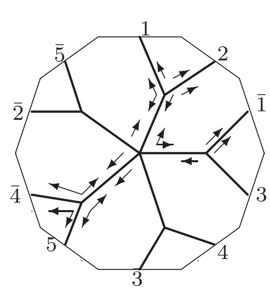
$W_6 : o2 - 01 - 20$



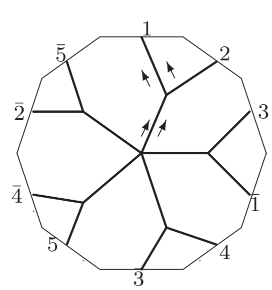
$W_6 : o2 - 02 - 20$



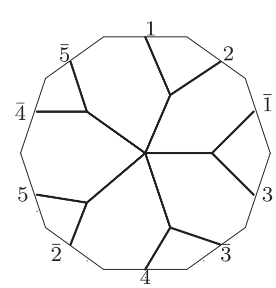
$W_6 : o3 - 03 - 20$



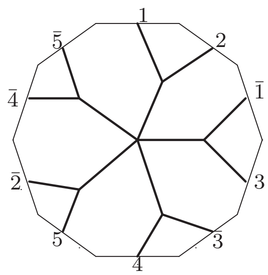
$W_6 : o2 - 04 - 20$



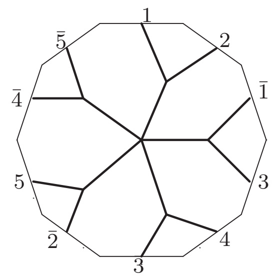
$W_6 : o2 - 05 - 04$



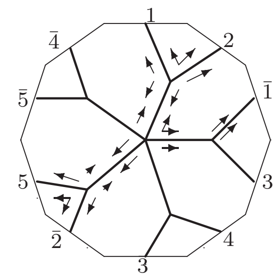
$W_6 : o2 - 06 - 40$



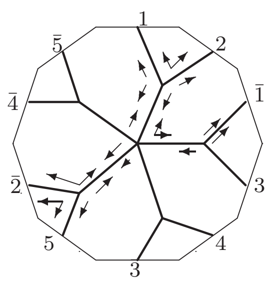
$W_6 : o2-07-40$



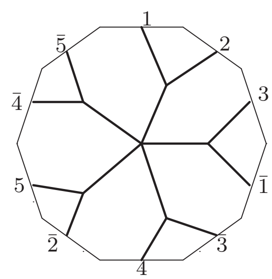
$W_6 : o2-08-40$



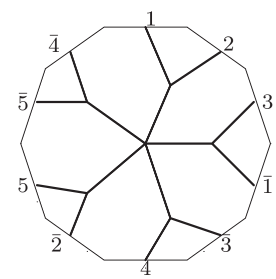
$W_6 : o2-09-20$



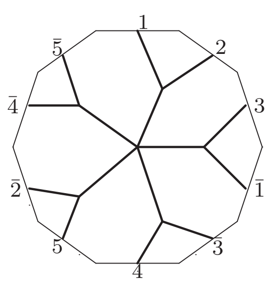
$W_6 : o2-10-20$



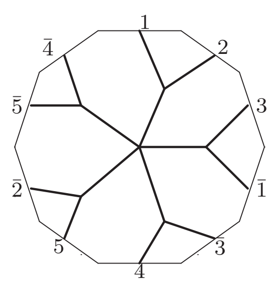
$W_6 : o2-11-40$



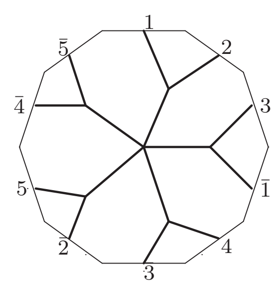
$W_6 : o2-12-40$



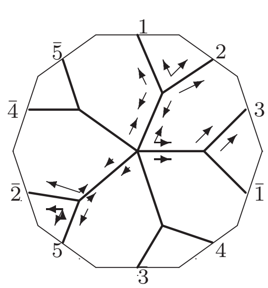
$W_6 : o2-13-40$



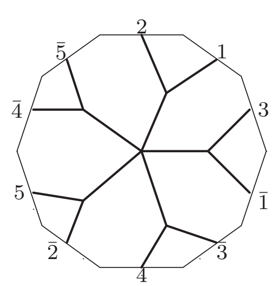
$W_6 : o2-14-40$



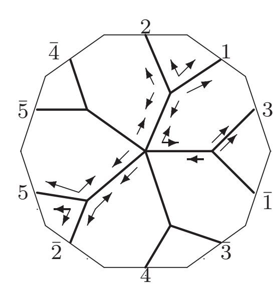
$W_6 : o2-15-40$



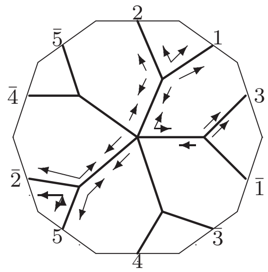
$W_6 : o2-16-20$



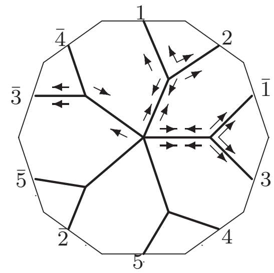
$W_6 : o2-17-40$



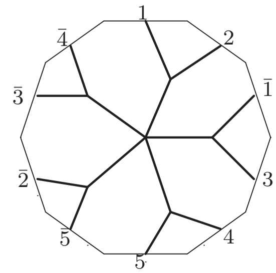
$W_6 : o2-18-20$



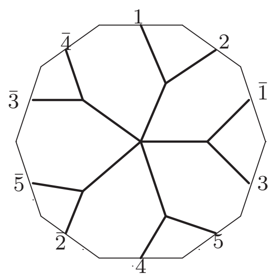
$W_6 : o2 - 19 - 20$



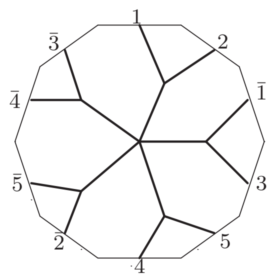
$W_6 : o2 - 20 - 20$



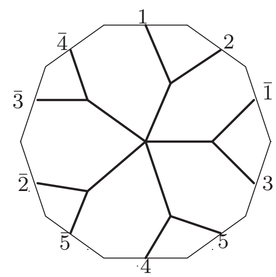
$W_6 : o2 - 21 - 40$



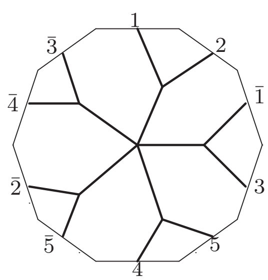
$W_6 : o2 - 22 - 40$



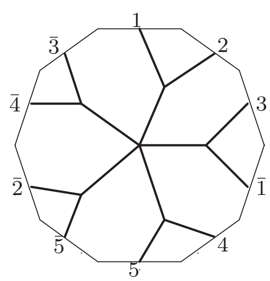
$W_6 : o2 - 23 - 40$



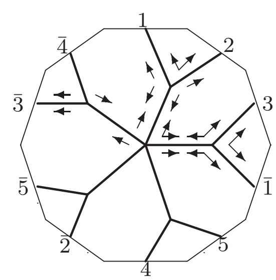
$W_6 : o2 - 24 - 40$



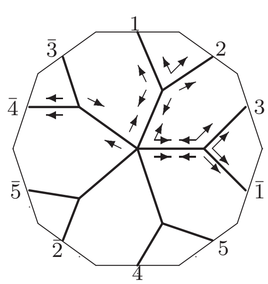
$W_6 : o2 - 25 - 40$



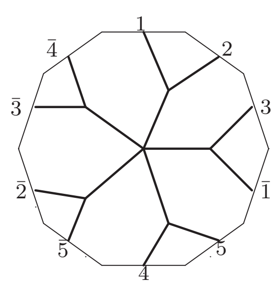
$W_6 : o2 - 26 - 40$



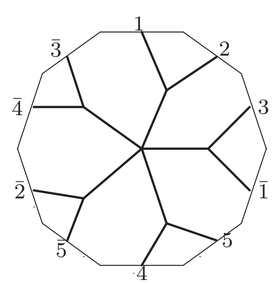
$W_6 : o2 - 27 - 20$



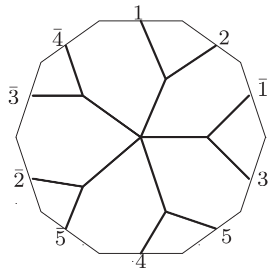
$W_6 : o2 - 28 - 20$



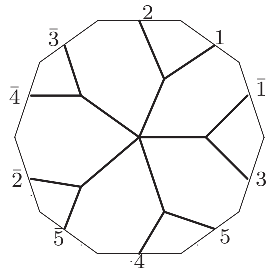
$W_6 : o2 - 29 - 40$



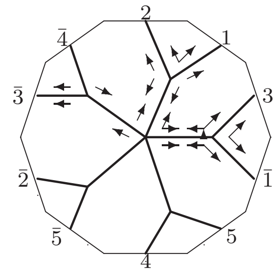
$W_6 : o2 - 30 - 40$



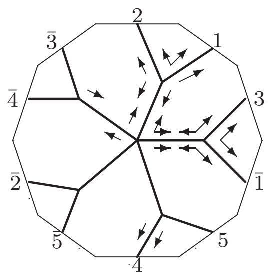
$W_6 : o2-31-40$



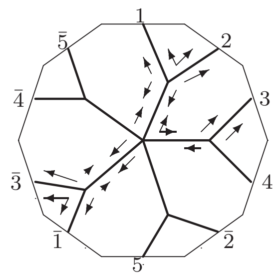
$W_6 : o2-32-40$



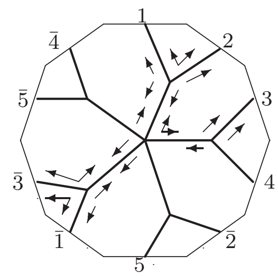
$W_6 : o2-33-20$



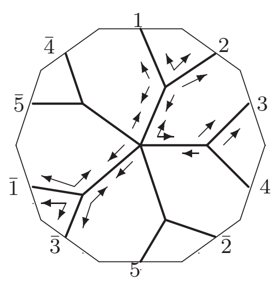
$W_6 : o2-34-20$



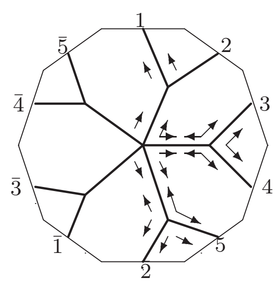
$W_6 : o2-35-20$



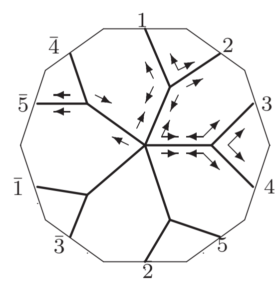
$W_6 : o2-36-20$



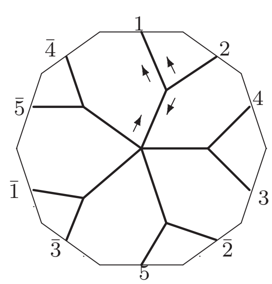
$W_6 : o2-37-20$



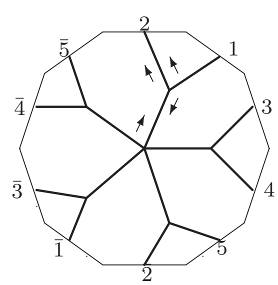
$W_6 : o2-38-20$



$W_6 : o2-39-20$



$W_6 : o2-40-04$



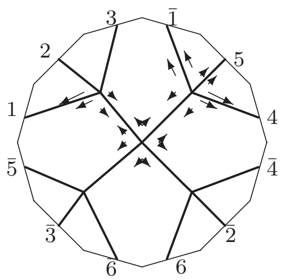
$W_6 : o2-41-04$

Ax.III.5 Complete graphs of order $5 \geq n \geq 4$

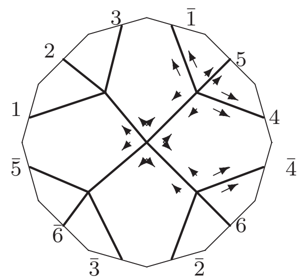
Size 4: ($K_4 = W_4$, W_4 is known in Ax.III.4).

Size 5:

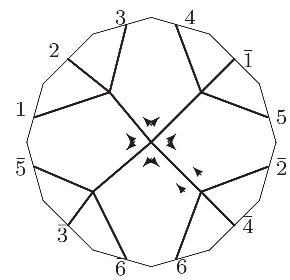
Orientable genus 1



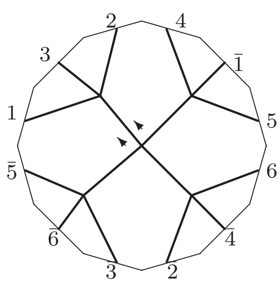
$K_5 : o1 - 01 - 20$



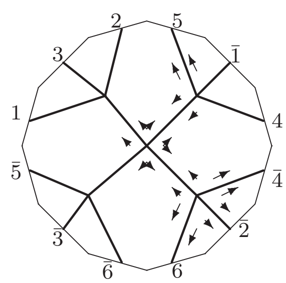
$K_5 : o1 - 02 - 20$



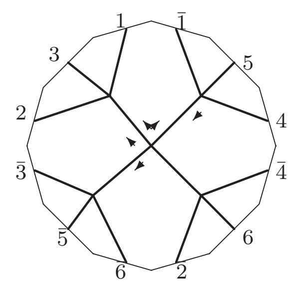
$K_5 : o1 - 03 - 10$



$K_5 : o1 - 04 - 02$

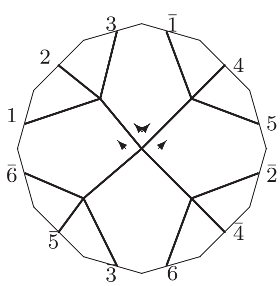


$W_6 : o1 - 05 - 20$

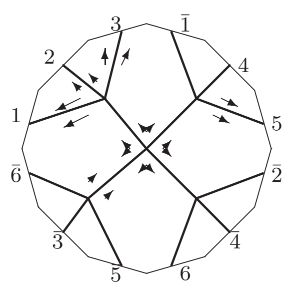


$W_6 : o1 - 06 - 05$

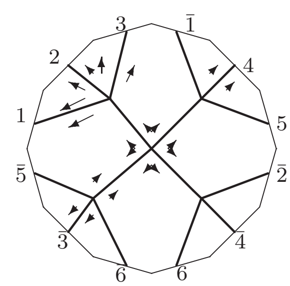
Orientable genus 2



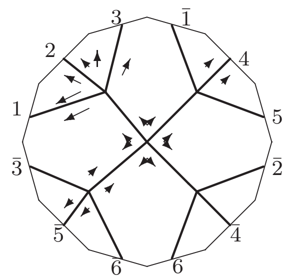
$K_5 : o2 - 01 - 04$

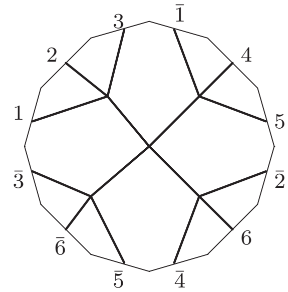


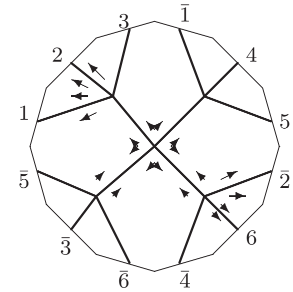
$K_5 : o2 - 02 - 20$



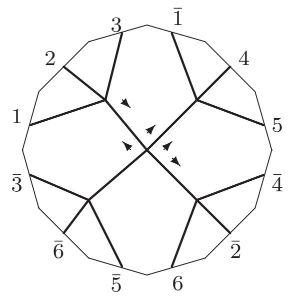
$K_5 : o2 - 03 - 20$

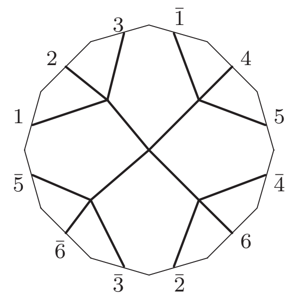


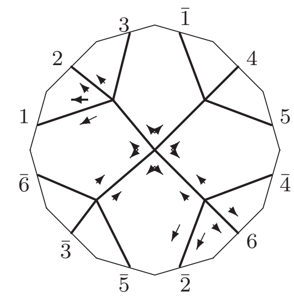




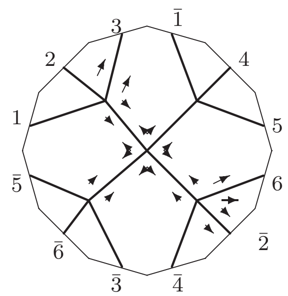
$K_5 : o2-04-20$ $K_5 : o2-05-40$ $K_5 : o2-06-20$

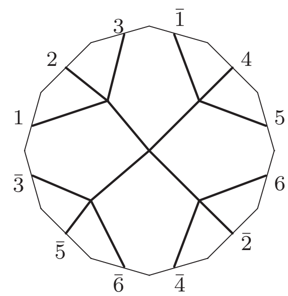


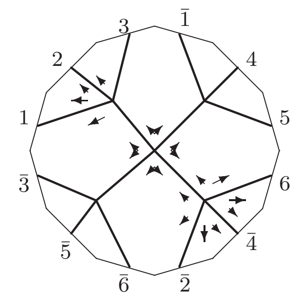




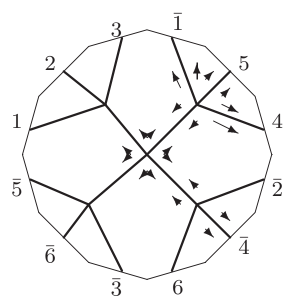
$K_5 : o2-07-05$ $K_5 : o2-08-40$ $K_5 : o2-09-20$

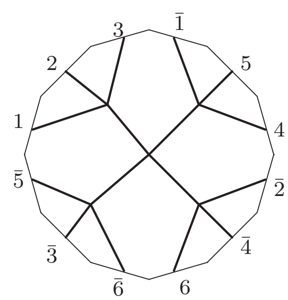


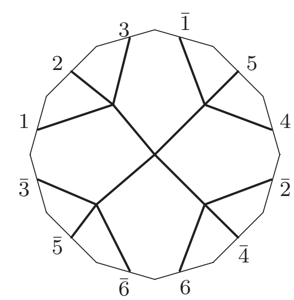




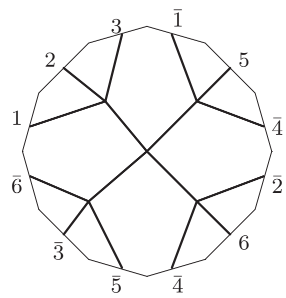
$K_5 : o2-10-20$ $K_5 : o2-11-40$ $K_5 : o2-12-20$



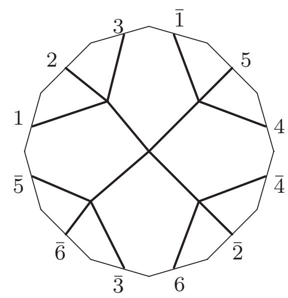




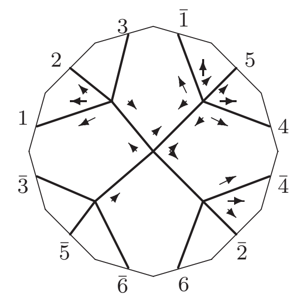
$K_5 : o2-13-20$ $K_5 : o2-14-40$ $K_5 : o2-15-40$



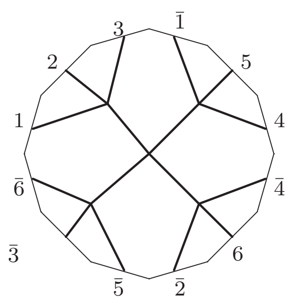
$K_5 : o2-16-40$



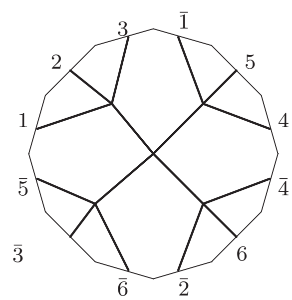
$K_5 : o2-17-40$



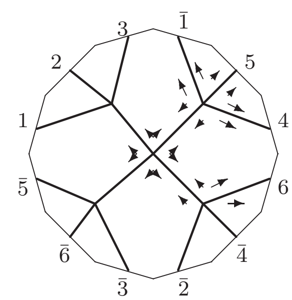
$K_5 : o2-18-20$



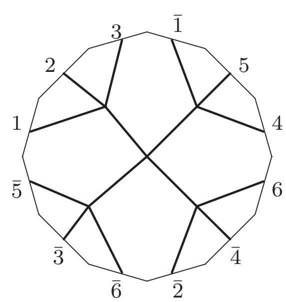
$K_5 : o2-19-40$



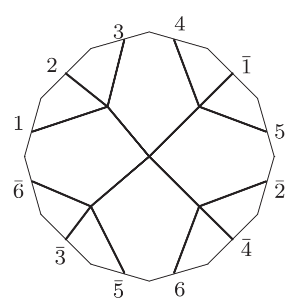
$K_5 : o2-20-40$



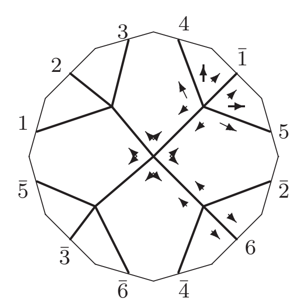
$K_5 : o2-21-20$



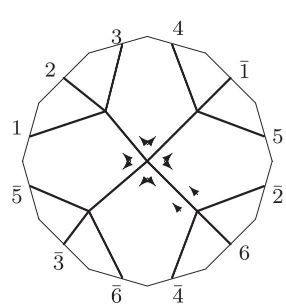
$K_5 : o2-22-40$



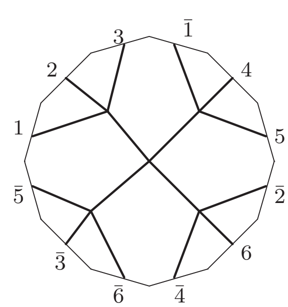
$K_5 : o2-23-40$



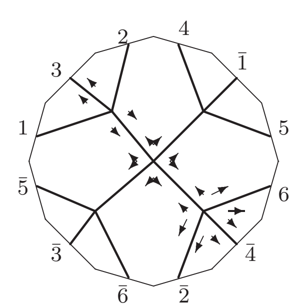
$K_5 : o2-24-20$



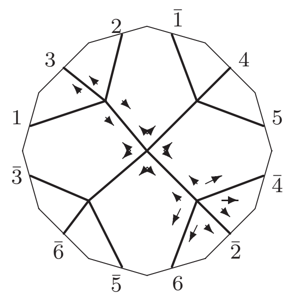
$K_5 : o2-25-10$



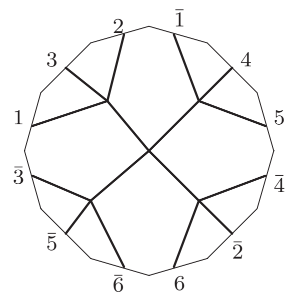
$K_5 : o2-26-40$



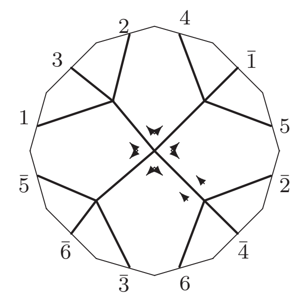
$K_5 : o2-27-20$



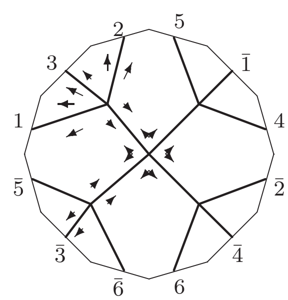
$K_5 : o2-28-20$



$K_5 : o2-29-40$

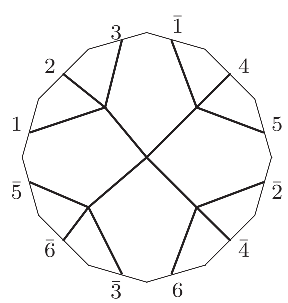


$K_5 : o2-30-10$

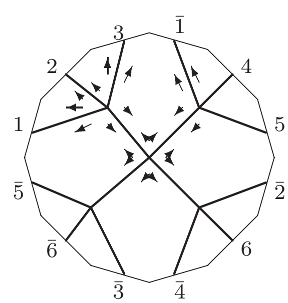


$K_5 : o2-31-20$

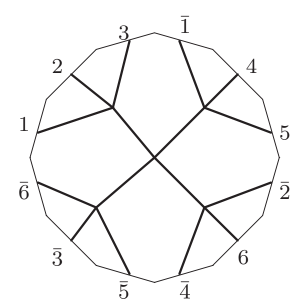
Orientable genus 3



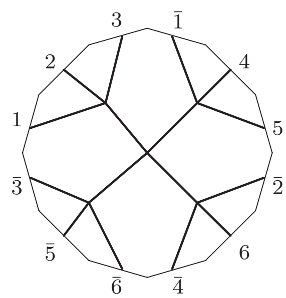
$K_5 : o3-01-40$



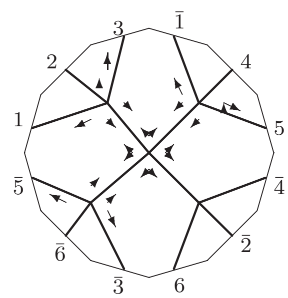
$K_5 : o3-02-20$



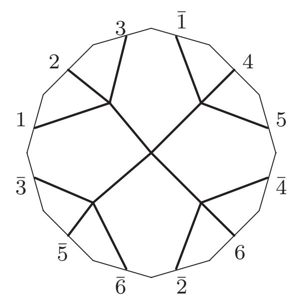
$K_5 : o3-03-40$



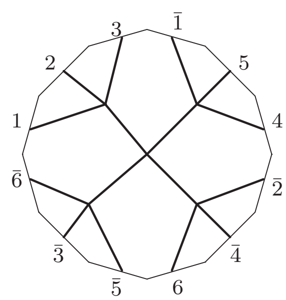
$K_5 : o3 - 04 - 40$



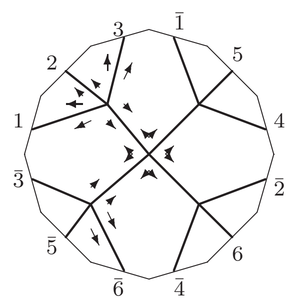
$K_5 : o3 - 05 - 20$



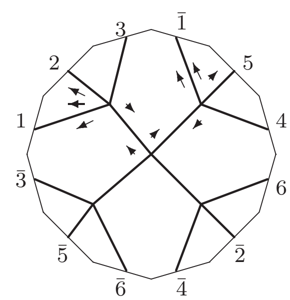
$K_5 : o3 - 06 - 40$



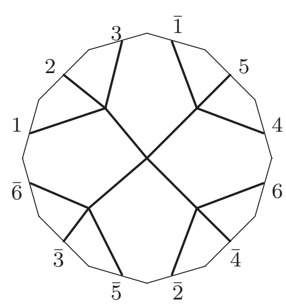
$K_5 : o3 - 07 - 40$



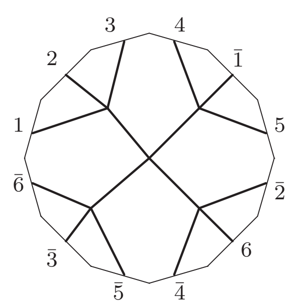
$K_5 : o3 - 08 - 20$



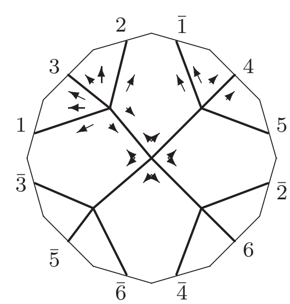
$K_5 : o3 - 09 - 10$



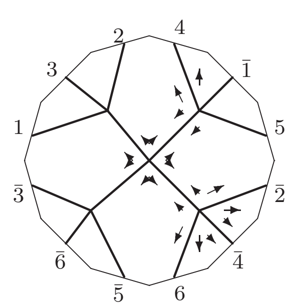
$K_5 : o3 - 10 - 40$



$K_5 : o3 - 11 - 40$



$K_5 : o3 - 12 - 20$



$K_5 : o3 - 13 - 20$

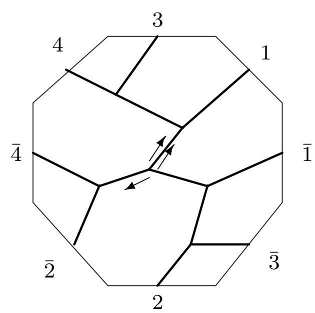
Ax.III.6 Triconnected cubic graphs of size in $[6, 15]$

Size 6: ($C_{6,1} = K_4 = W_4$, W_4 is known above).

Size 9: ($C_{9,2} = K_{3,3}$)

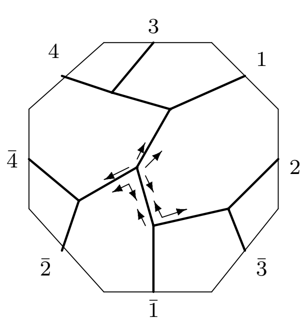
$C_{9,1}$:

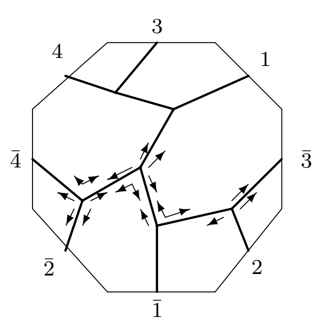
Orientable genus 0

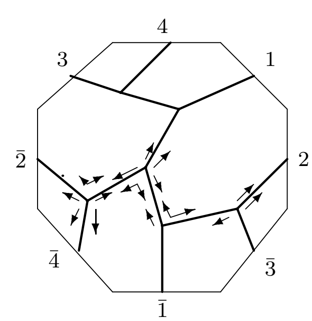


$C_{9,1} : o0 - 01 - 03$

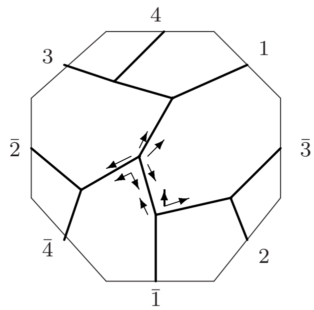
Orientable genus 1

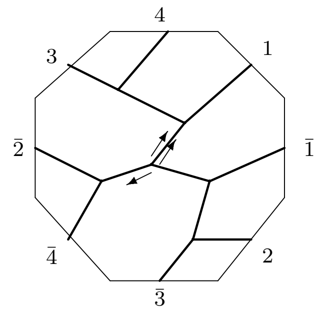






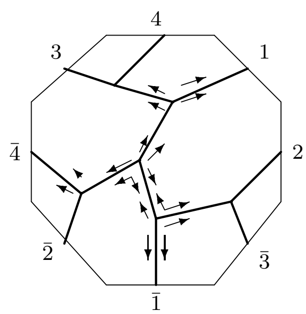
$C_{9,1} : o1 - 01 - 09$ $C_{9,1} : o1 - 02 - 18$ $C_{9,1} : o1 - 03 - 18$

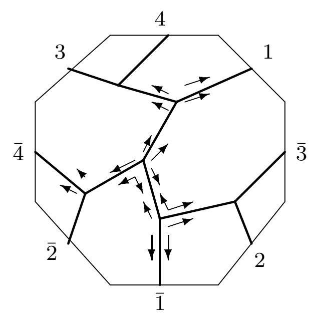




$C_{9,1} : o1 - 04 - 09$ $\qquad$ $C_{9,1} : o1 - 05 - 18$

Orientable genus 2

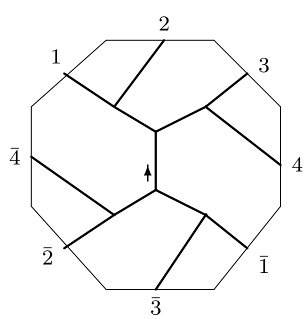


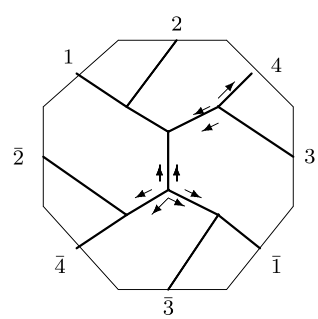


$C_{9,1} : o2 - 01 - 18$ $\qquad$ $C_{9,1} : o2 - 02 - 18$

$C_{9,2}$:

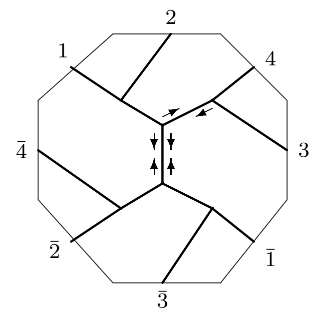
Orientable genus 1





$C_{9,2} : o1 - 01 - 01$ $\qquad$ $C_{9,2} : o1 - 02 - 09$

Orientable genus 2

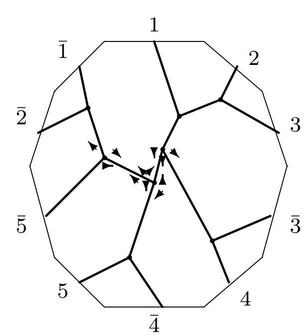


$C_{9,2} : o2 - 01 - 06$

Size 12: ($C_{12,4}$ is the cube)

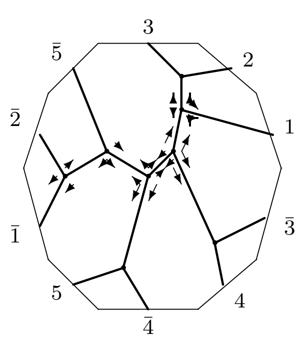
$C_{12,1}$:

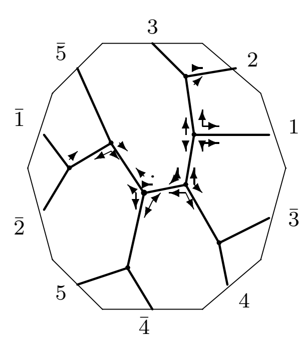
Orientable genus 0

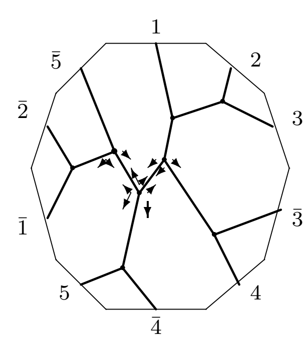


$C_{12,1} : o0 - 01 - 12$

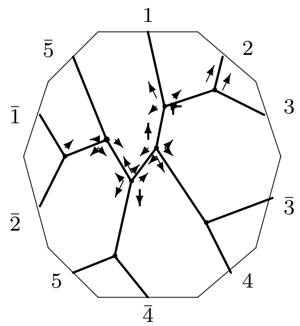
Orientable genus 1



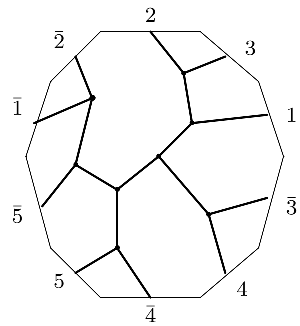




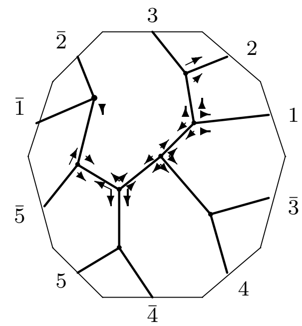
$C_{12,1} : o1 - 01 - 24$ $C_{12,1} : o1 - 02 - 24$ $C_{12,1} : o1 - 03 - 12$



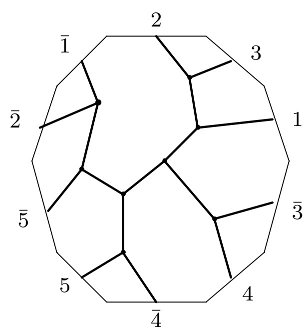
$C_{12,1} : o1 - 04 - 24$



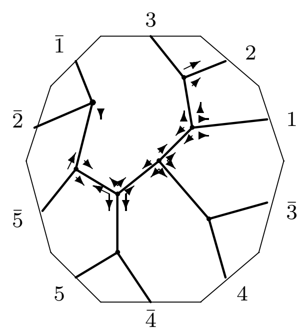
$C_{12,1} : o1 - 05 - 48$



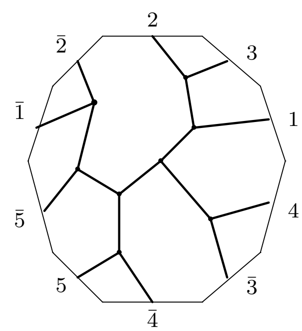
$C_{12,1} : o1 - 06 - 24$



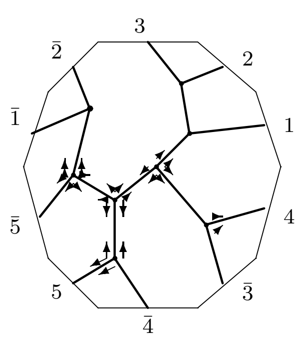
$C_{12,1} : o1 - 07 - 48$



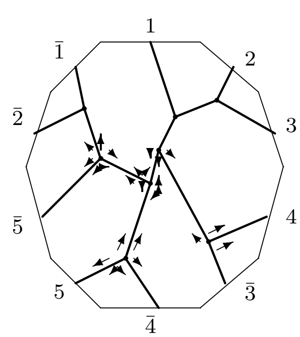
$C_{12,1} : o1 - 08 - 24$



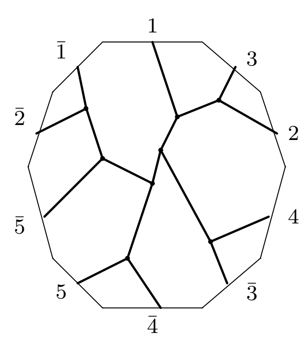
$C_{12,1} : o1 - 09 - 48$



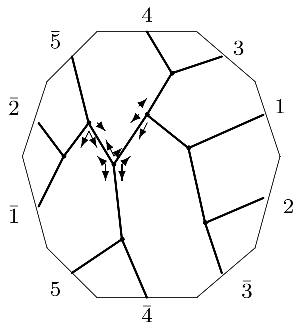
$C_{12,1} : o1 - 10 - 24$



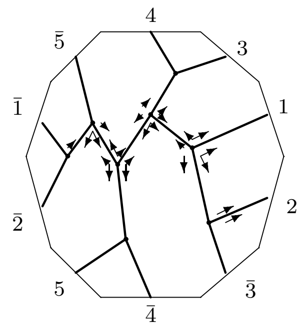
$C_{12,1} : o1 - 11 - 24$



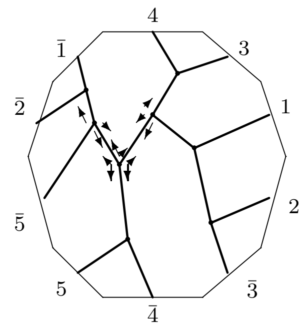
$C_{12,1} : o1 - 12 - 48$



$C_{12,1} : o1 - 13 - 12$

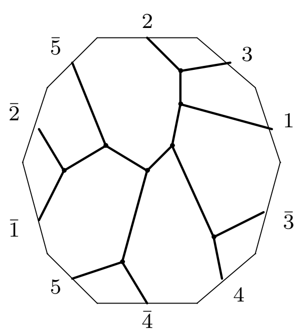


$C_{12,1} : o1 - 14 - 24$

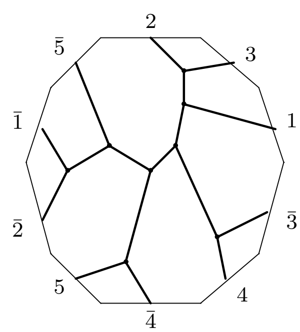


$C_{12,1} : o1 - 15 - 12$

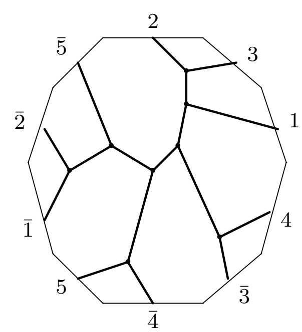
Orientable genus 2



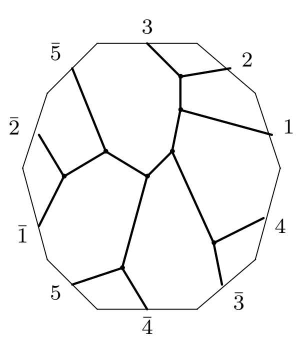
$C_{12,1} : o2 - 01 - 28$



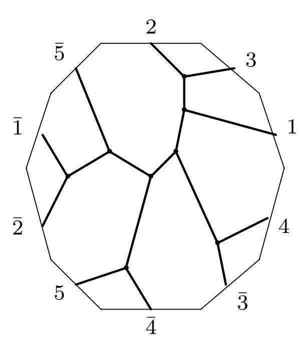
$C_{12,1} : o2 - 02 - 48$



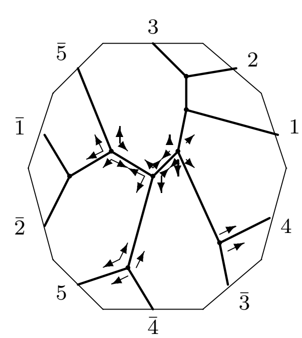
$C_{12,1} : o2 - 03 - 48$



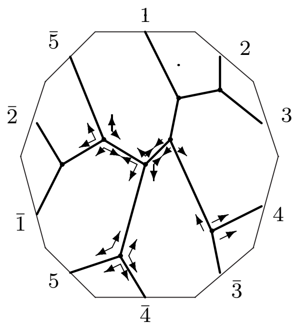
$C_{12,1} : o2 - 04 - 48$



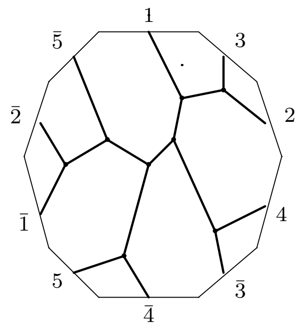
$C_{12,1} : o2 - 05 - 48$



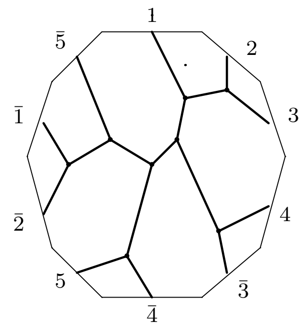
$C_{12,1} : o2 - 06 - 24$



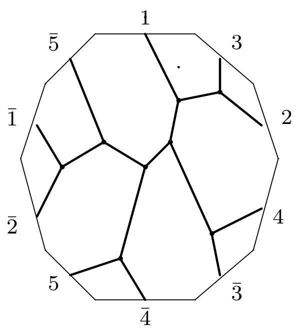
$C_{12,1} : o2 - 07 - 24$



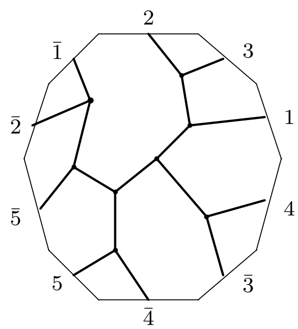
$C_{12,1} : o2 - 08 - 24$



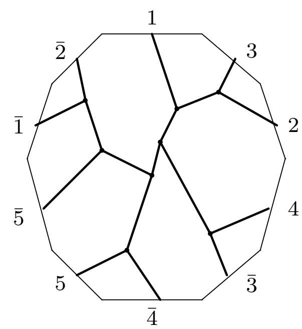
$C_{12,1} : o2 - 09 - 48$



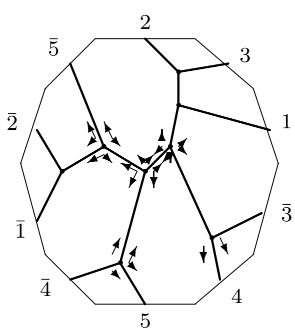
$C_{12,1} : o2 - 10 - 48$



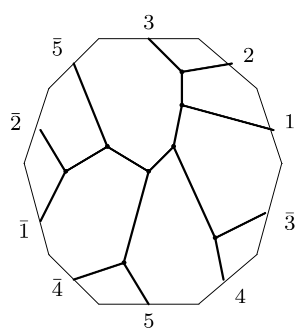
$C_{12,1} : o2 - 11 - 48$



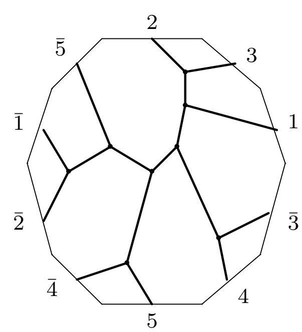
$C_{12,1} : o2 - 12 - 48$



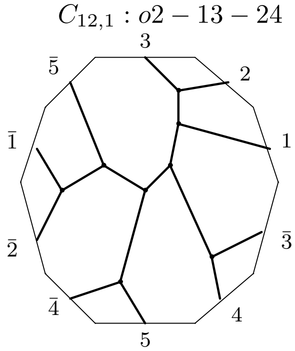
$C_{12,1} : o2 - 13 - 24$



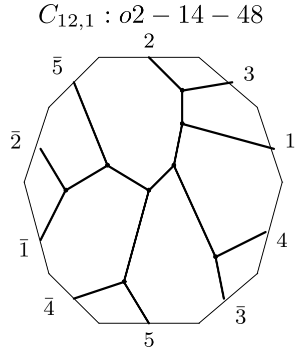
$C_{12,1} : o2 - 14 - 48$



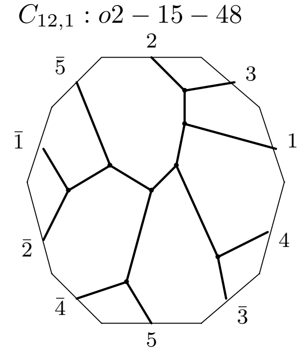
$C_{12,1} : o2 - 15 - 48$



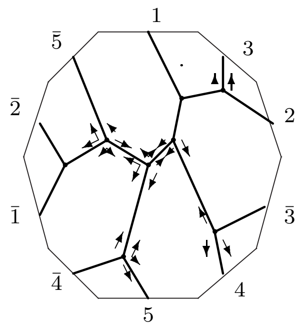
$C_{12,1} : o2 - 16 - 48$



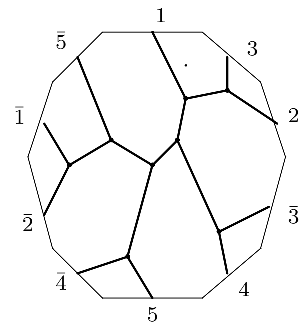
$C_{12,1} : o2 - 17 - 48$



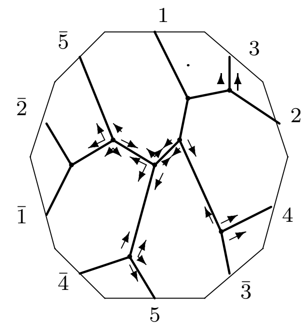
$C_{12,1} : o2 - 18 - 48$



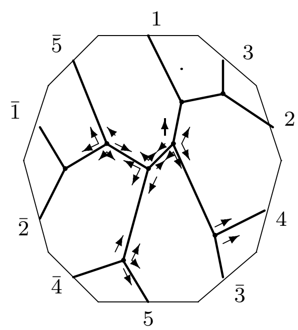
$C_{12,1} : o2 - 19 - 24$



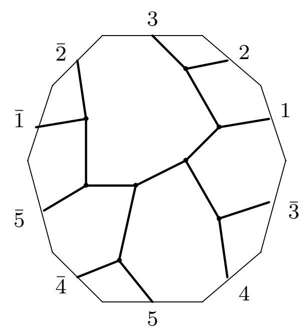
$C_{12,1} : o2 - 20 - 48$



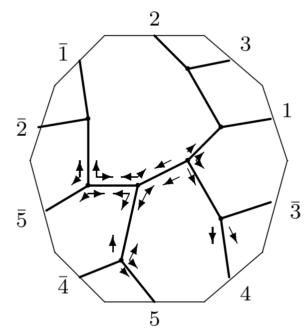
$C_{12,1} : o2 - 21 - 24$



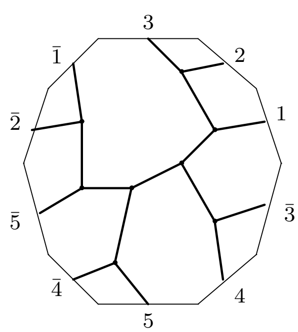
$C_{12,1} : o2 - 22 - 24$



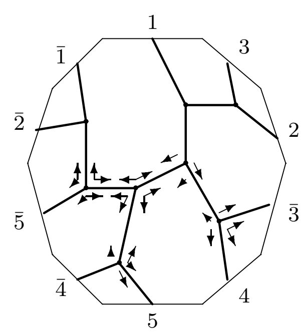
$C_{12,1} : o2 - 23 - 48$



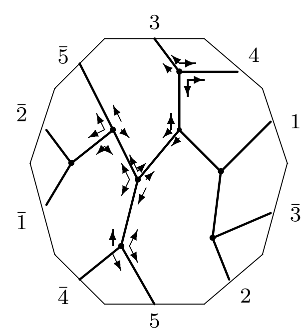
$C_{12,1} : o2 - 24 - 24$



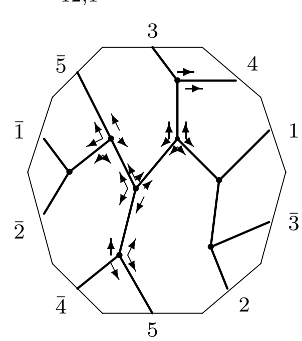
$C_{12,1} : o2 - 25 - 48$



$C_{12,1} : o2 - 26 - 24$



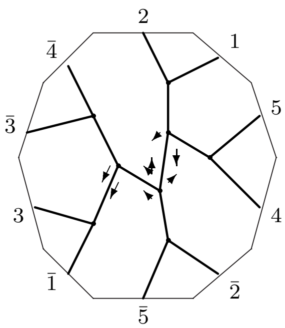
$C_{12,1} : o2 - 27 - 24$



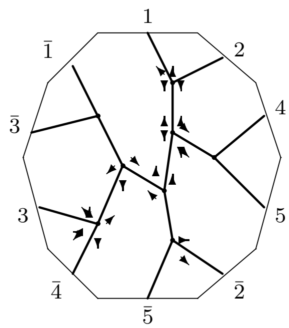
$C_{12,1} : o2 - 28 - 24$

$C_{12,2}$:

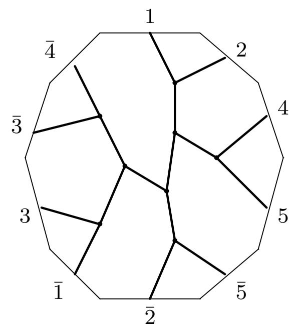
Orientable genus 1



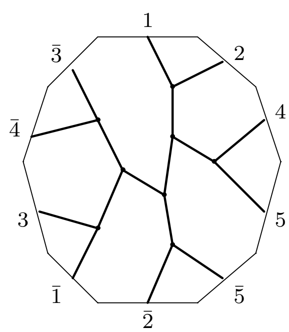
$C_{12,2} : o1 - 01 - 08$



$C_{12,2} : o1 - 02 - 24$

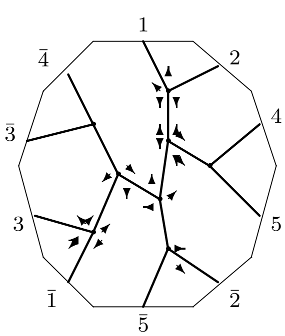


$C_{12,2} : o1 - 03 - 48$

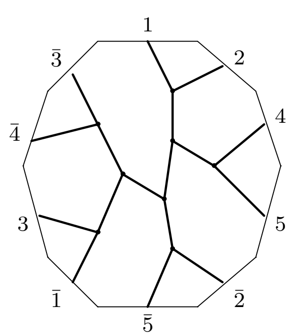


$C_{12,2} : o1 - 04 - 48$

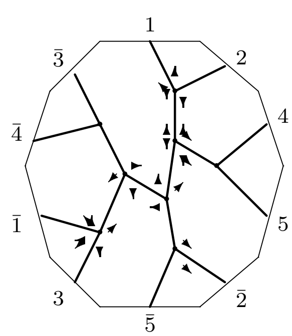
Orientable genus 2



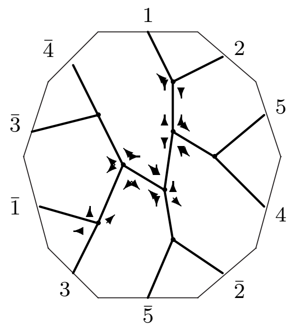
$C_{12,2} : o2 - 01 - 24$



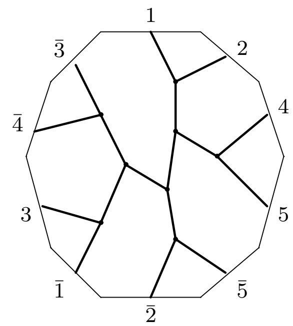
$C_{12,2} : o2 - 02 - 48$



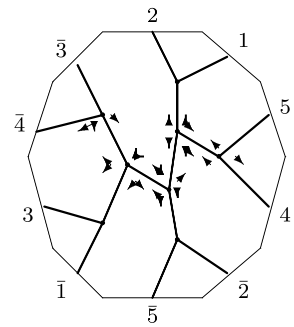
$C_{12,2} : o2 - 03 - 24$



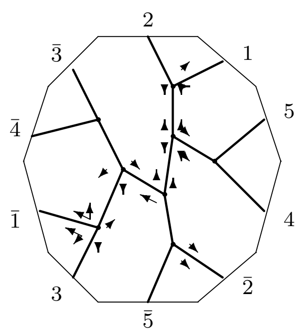
$C_{12,2} : o2-04-24$



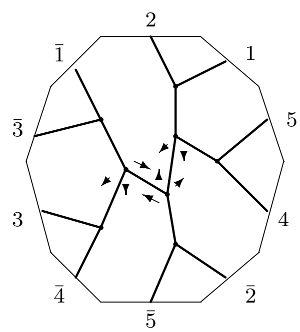
$C_{12,2} : o2-05-48$



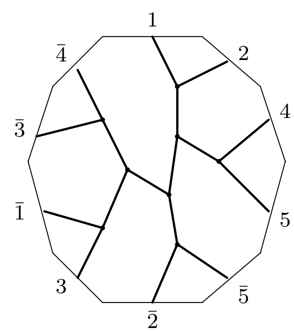
$C_{12,2} : o2-06-24$



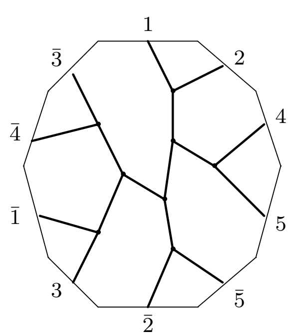
$C_{12,2} : o2-07-24$



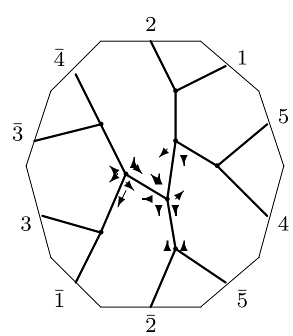
$C_{12,2} : o2-08-08$



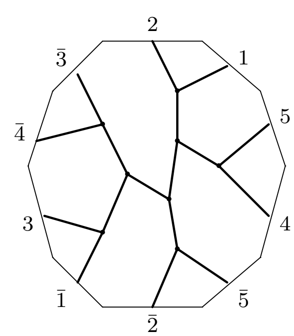
$C_{12,2} : o2-09-48$



$C_{12,2} : o2-10-48$



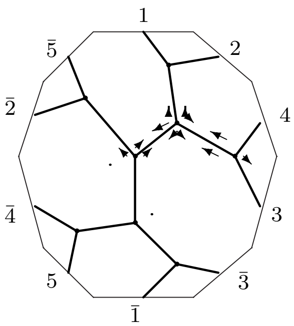
$C_{12,2} : o2-11-16$

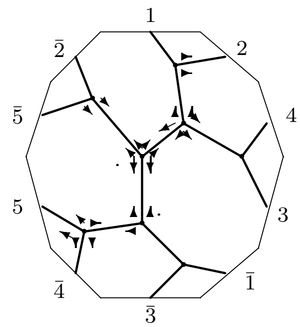


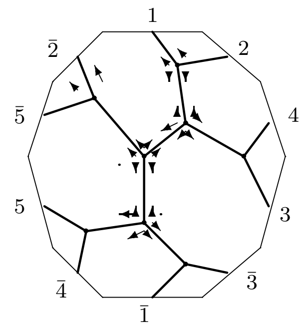
$C_{12,2} : o2-12-48$

$C_{12,3}$:

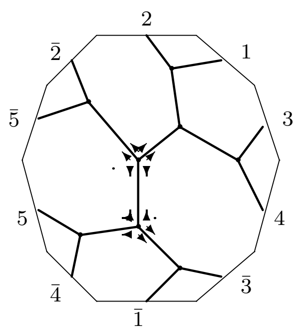
Orientable genus 1

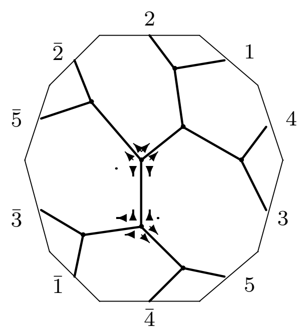






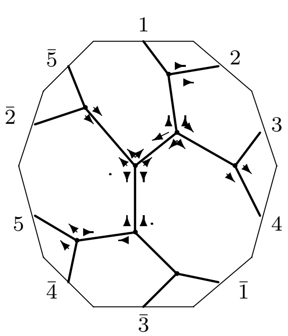
$C_{12,3} : o1 - 01 - 12$ $C_{12,3} : o1 - 02 - 24$ $C_{12,3} : o1 - 03 - 24$

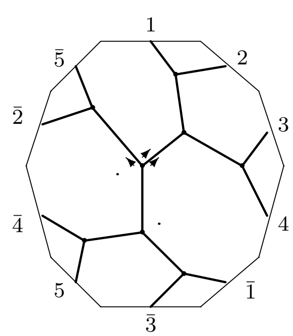


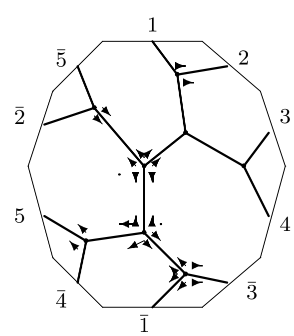


$C_{12,3} : o1 - 04 - 12$ $C_{12,3} : o1 - 05 - 12$

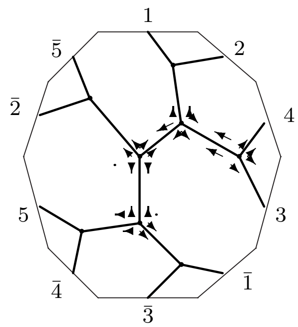
Orientable genus 2



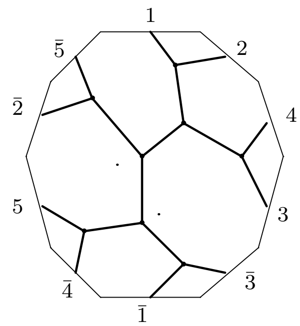




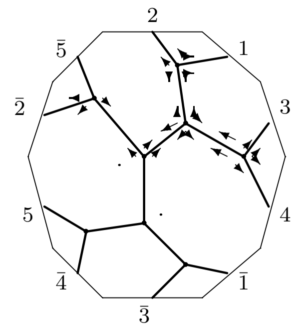
$C_{12,3} : o2 - 01 - 24$ $C_{12,3} : o2 - 02 - 03$ $C_{12,3} : o2 - 03 - 24$



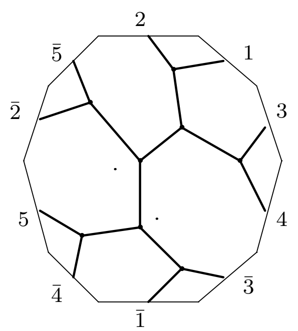
$C_{12,3} : o2-04-24$



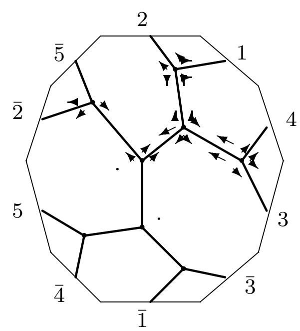
$C_{12,3} : o2-05-48$



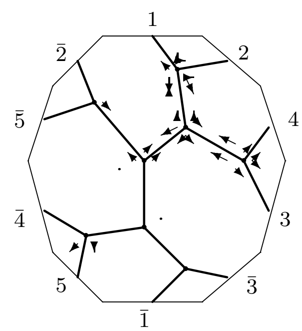
$C_{12,3} : o2-06-24$



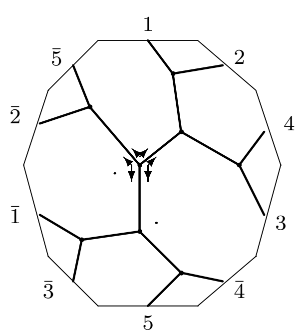
$C_{12,3} : o2-07-48$



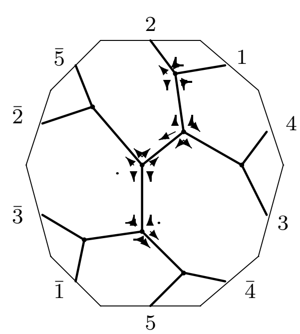
$C_{12,3} : o2-08-24$



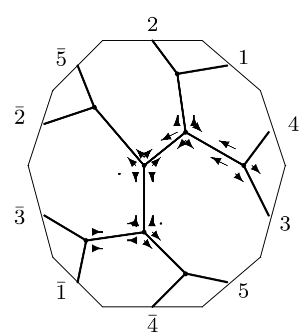
$C_{12,3} : o2-09-24$



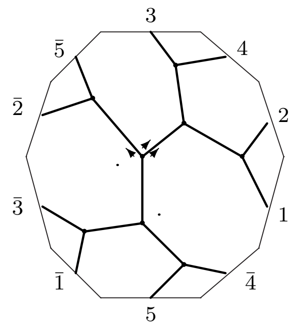
$C_{12,3} : o2-10-06$



$C_{12,3} : o2-11-24$



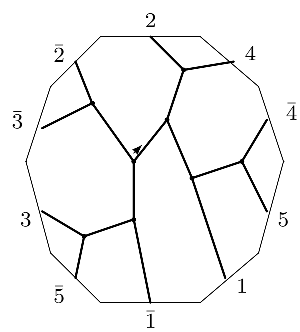
$C_{12,3} : o2-12-24$



$C_{12,3} : o2 - 13 - 03$

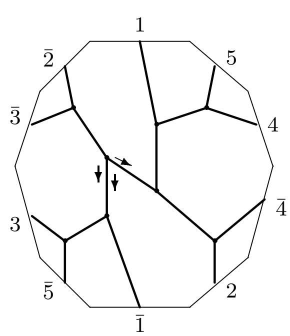
$C_{12,4}$:

Orientable genus 0

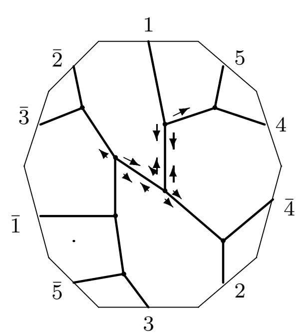


$C_{12,4} : o0 - 01 - 01$

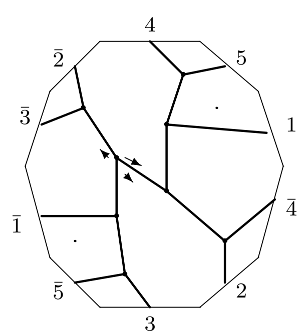
Orientable genus 1



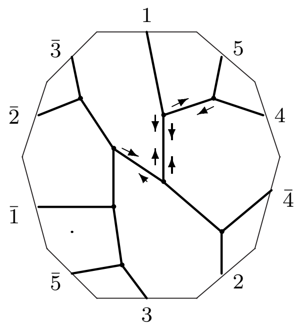
$C_{12,4} : o1 - 01 - 03$



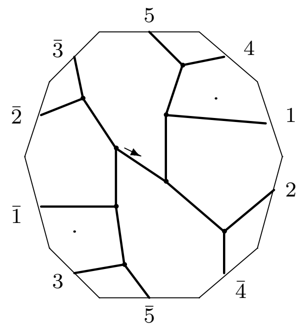
$C_{12,4} : o1 - 02 - 12$



$C_{12,4} : o1 - 03 - 03$

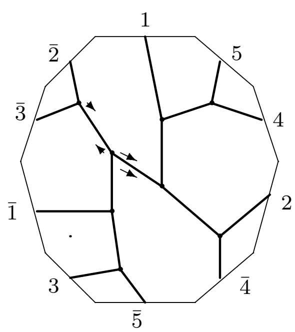


$C_{12,4} : o1 - 04 - 08$

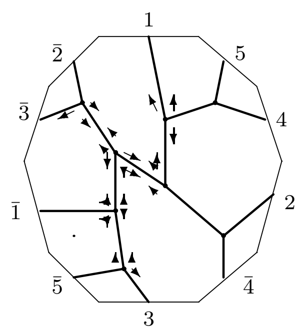


$C_{12,4} : o1 - 05 - 01$

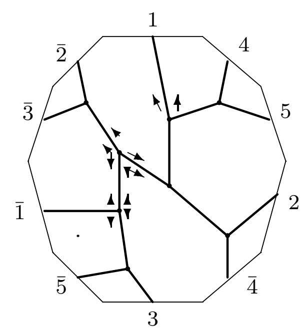
Orientable genus 2



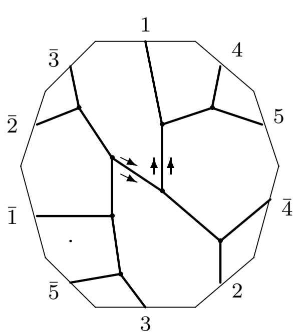
$C_{12,4} : o2 - 01 - 04$



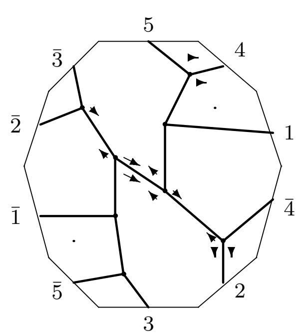
$C_{12,4} : o2 - 02 - 24$



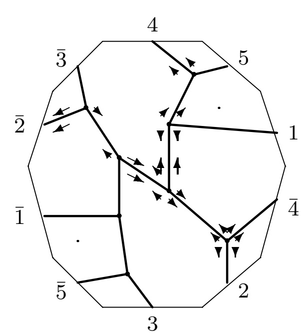
$C_{12,4} : o2 - 03 - 12$



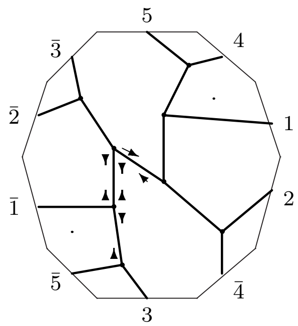
$C_{12,4} : o2 - 04 - 04$

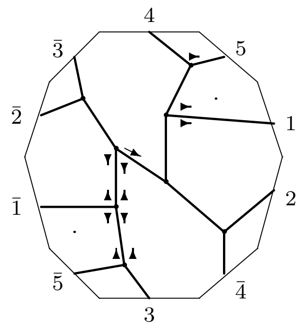


$C_{12,4} : o2 - 05 - 12$



$C_{12,4} : o2 - 06 - 24$



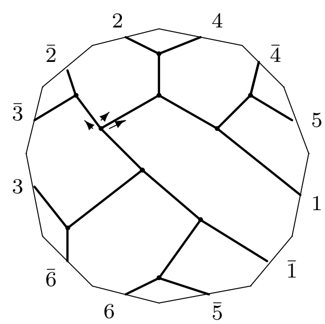


$C_{12,4} : o2 - 07 - 08$ $C_{12,4} : o2 - 08 - 12$

Size 15: Two chosen where $C_{15,14}$ is the Petersen graph.

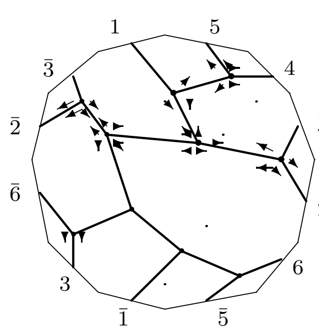
$C_{15,11}$:

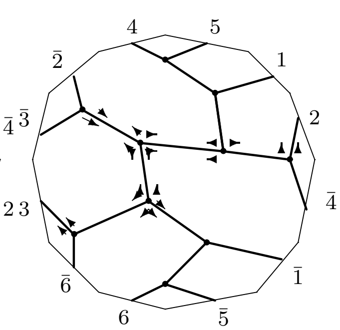
Orientable genus 0

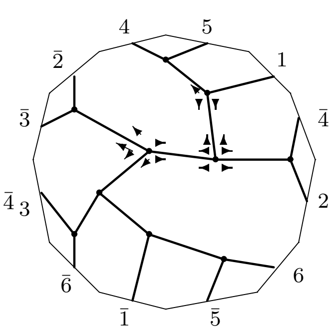


$C_{15,11} : o0 - 01 - 03$

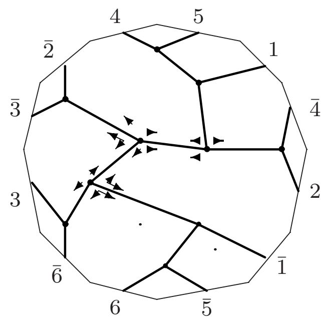
Orientable genus 1



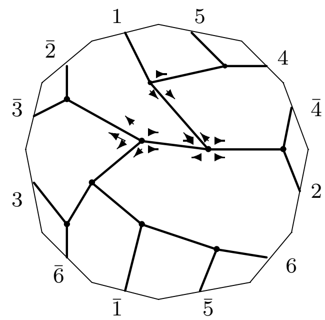




$C_{15,11} : o1 - 01 - 30$ $C_{15,11} : o1 - 02 - 30$ $C_{15,11} : o2 - 03 - 15$

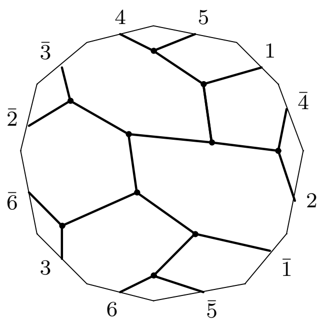


$C_{15,11} : o1 - 04 - 15$

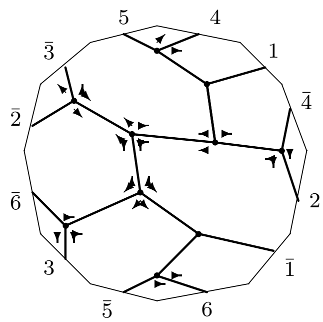


$C_{15,11} : o1 - 05 - 15$

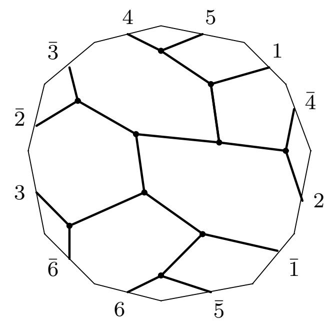
Orientable genus 2



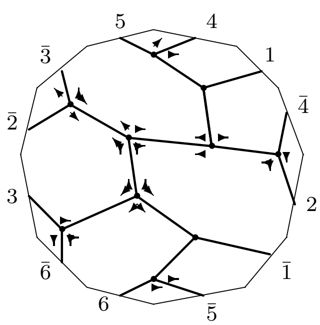
$C_{15,11} : o2 - 01 - 60$



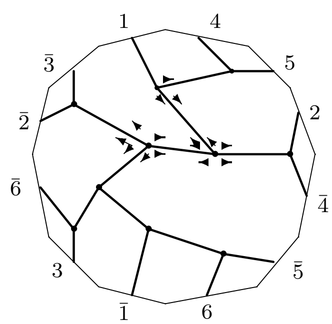
$C_{15,11} : o2 - 02 - 30$



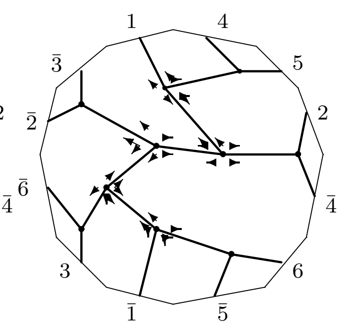
$C_{15,11} : o2 - 03 - 60$



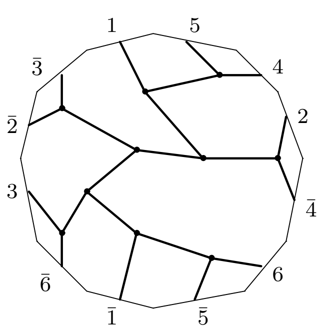
$C_{15,11} : o2 - 04 - 30$



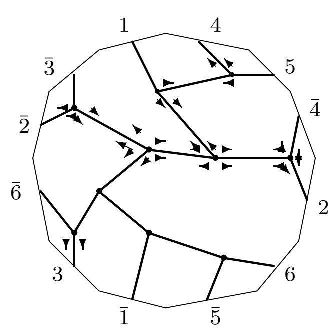
$C_{15,11} : o2 - 05 - 15$



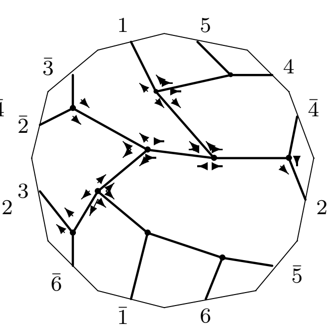
$C_{15,11} : o2 - 06 - 30$



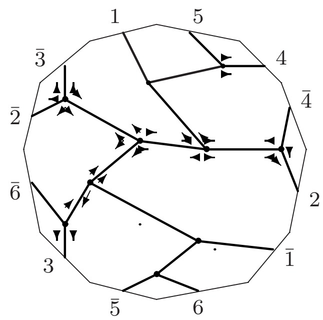
$C_{15,11} : o2 - 07 - 60$

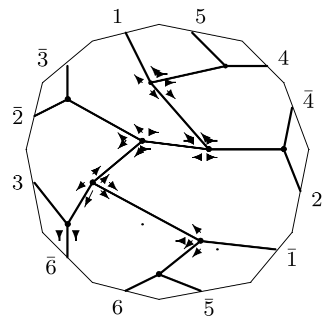


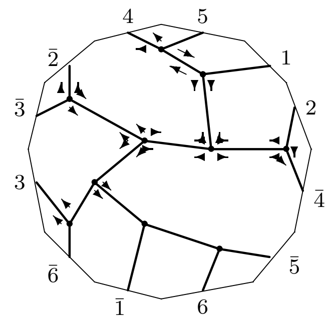
$C_{15,11} : o2 - 08 - 30$



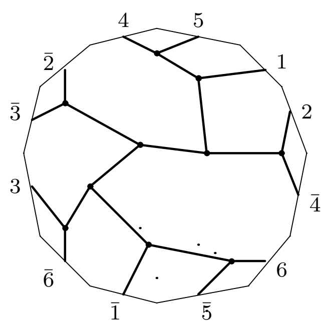
$C_{15,11} : o2 - 09 - 30$

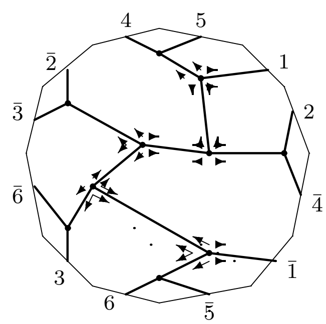


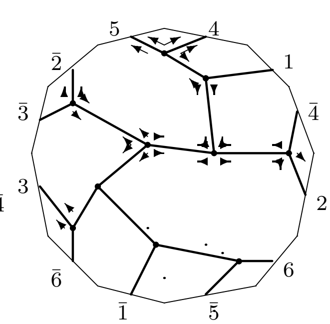




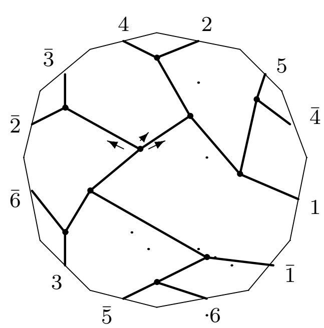
$C_{15,11} : o2-10-30$ $C_{15,11} : o2-11-30$ $C_{15,11} : o2-12-30$

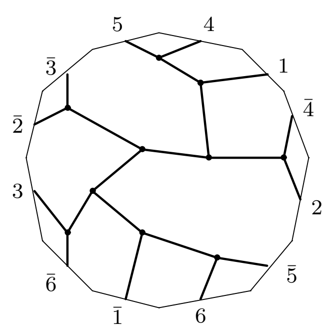


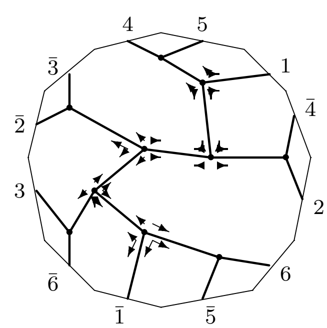




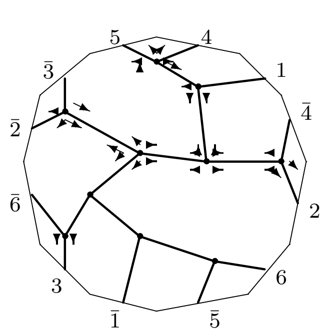
$C_{15,11} : o2-13-60$ $C_{15,11} : o2-14-30$ $C_{15,11} : o2-15-30$

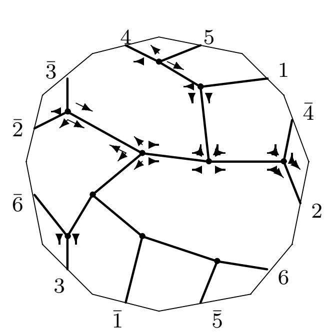


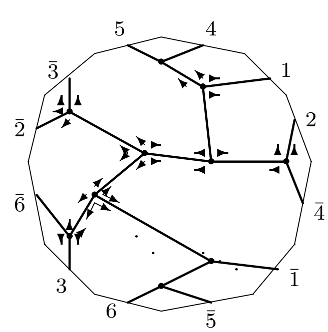




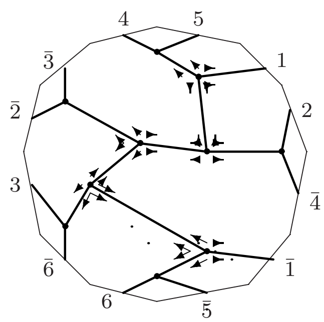
$C_{15,11} : o2-16-03$ $C_{15,11} : o2-17-60$ $C_{15,11} : o2-18-30$



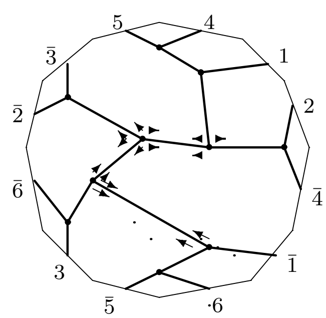




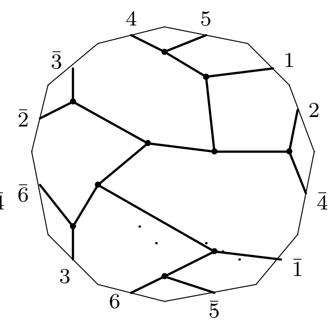
$C_{15,11} : o2-19-30$ $C_{15,11} : o2-20-30$ $C_{15,11} : o2-21-30$



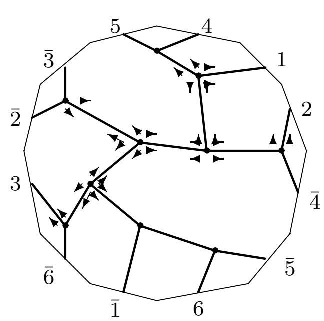
$C_{15,11}: o2-22-30$



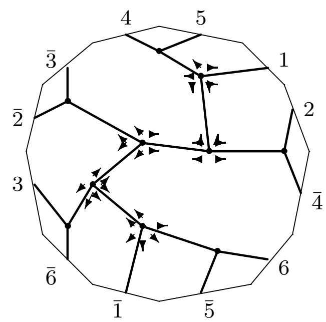
$C_{15,11}: o2-23-15$



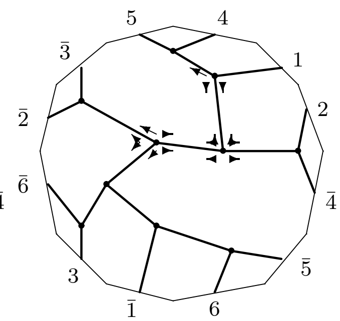
$C_{15,11}: o2-24-60$



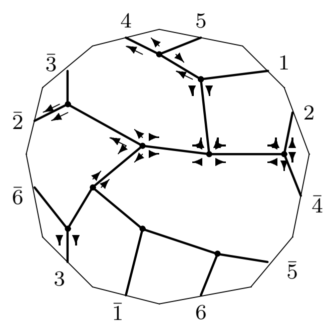
$C_{15,11}: o2-25-30$



$C_{15,11}: o2-26-30$

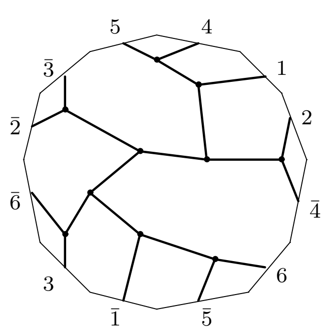


$C_{15,11}: o2-27-15$

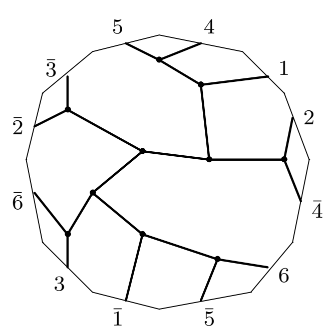


$C_{15,11}: o2-28-30$

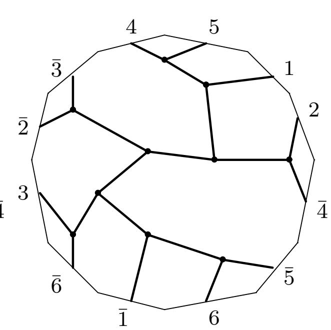
Orientable genus 3



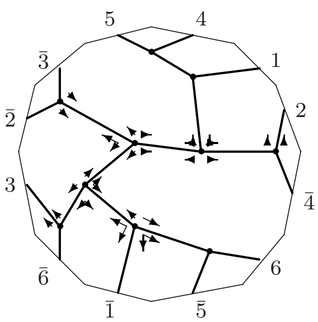
$C_{15,11}: o3-01-60$



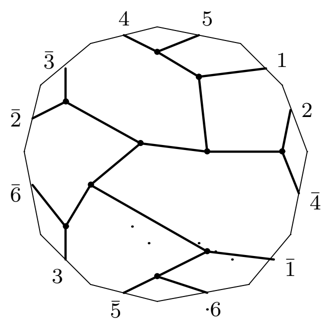
$C_{15,11}: o3-02-60$



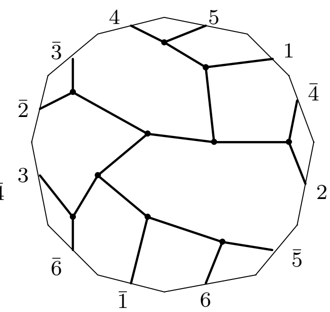
$C_{15,11}: o3-03-60$



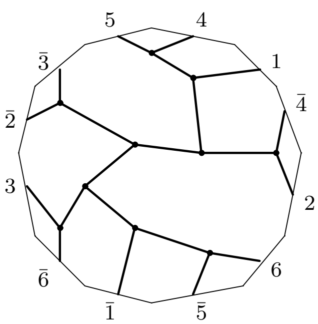
$C_{15,11} : o3 - 04 - 30$



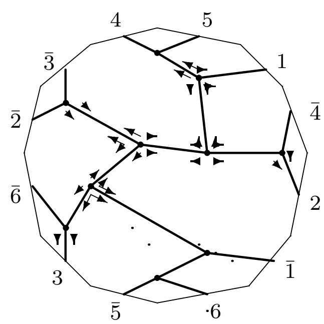
$C_{15,11} : o3 - 05 - 60$



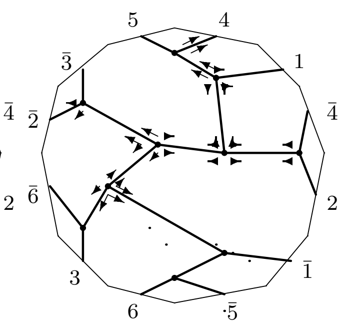
$C_{15,11} : o3 - 06 - 60$



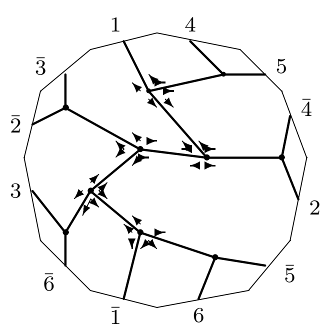
$C_{15,11} : o3 - 07 - 60$



$C_{15,11} : o3 - 08 - 30$



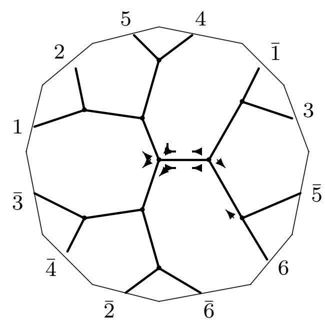
$C_{15,11} : o3 - 09 - 30$



$C_{15,11} : o3 - 10 - 30$

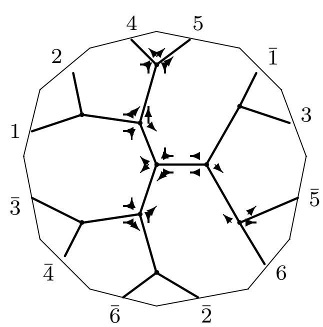
$C_{15,14}$:

Orientable genus 1

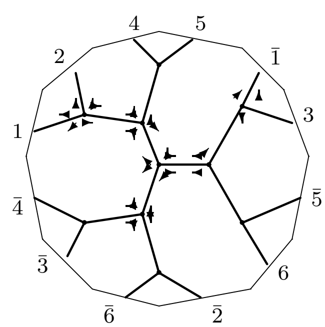


$C_{15,14} : o1 - 01 - 10$

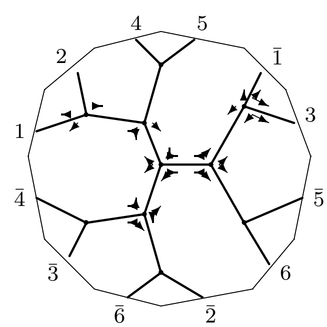
Orientable genus 2



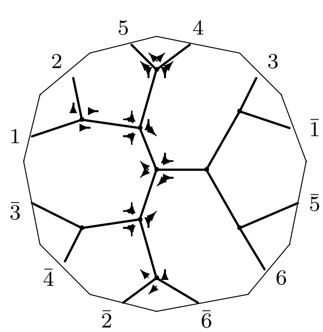
$C_{15,14} : o2 - 01 - 30$



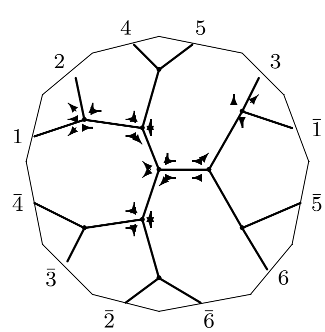
$C_{15,14} : o2 - 02 - 30$



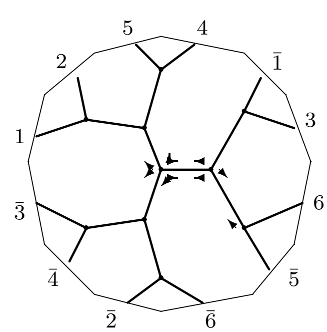
$C_{15,14} : o2 - 03 - 30$



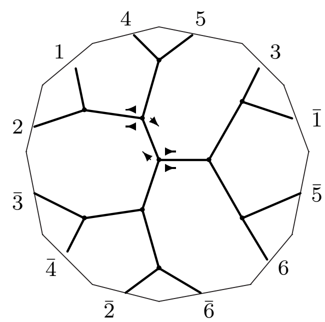
$C_{15,14} : o2 - 04 - 20$



$C_{15,14} : o2 - 05 - 30$

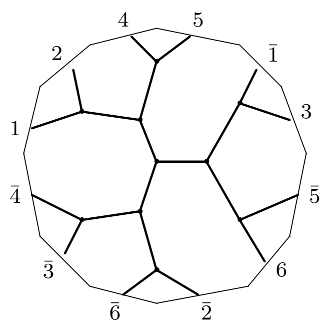


$C_{15,14} : o2 - 06 - 10$

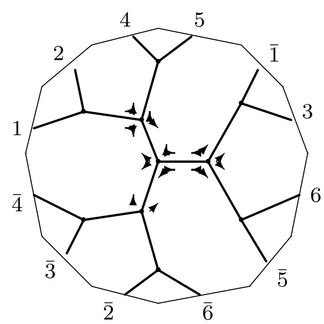


$C_{15,14} : o2 - 07 - 06$

Orientable genus 3



$C_{15,14} : o3 - 01 - 60$



$C_{15,14} : o3 - 02 - 20$

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Terminology

Abstract: A *Smarandache system* $(\Sigma; \mathcal{R})$ is such a mathematical system with at least one Smarandachely denied rule $\overline{r}$ in $\mathcal{R}$ such that it behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways. A map is a 2-cell decomposition of surface, which can be seen as a connected graphs in development from partition to permutation, also a basis for constructing Smarandache systems, particularly, Smarandache 2-manifolds for Smarandache geometry. As an introductory book, this book contains the elementary materials in map theory, including embeddings of a graph, abstract maps, duality, orientable and non-orientable maps, isomorphisms of maps and the enumeration of rooted or unrooted maps, particularly, the *joint tree* representation of an embedding of a graph on two dimensional manifolds, which enables one to make the complication much simpler on map enumeration. All of these are valuable for researchers and students in combinatorics, graphs and low dimensional topology.

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